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Quark Confinement in Light-Front QCD and A Weak-Coupling Treatment to Heavy Hadrons

Wei-Min Zhang*

Institute of Physics, Academia Sinica, Taipei 11529, Taiwan

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Abstract

In this paper, we develop a weak-coupling treatment of nonperturbative QCD to heavy hadrons on the light-front. First, we present a derivation of quark confining interaction in light-front QCD for heavy quark systems, based on the recently developed light-front similarity renormalization group approach and the light-front heavy quark effective theory. The resulting effective light-front QCD Hamiltonian H_λ at a low-energy cutoff λ manifests the coexistence of a confining potential and a Coulomb potential. A clear light-front picture of quark confinement emerges. Using this low energy QCD Hamiltonian H_λ, we study heavy hadron bound state equations in the framework of a recently proposed possible weak-coupling treatment of non-perturbative QCD. Light-front heavy hadron bound states with definite spin and parity are constructed and the general structure of the corresponding wavefunctions is explored. A Gaussian-type wavefunction ansatz is used to solve the light-front quarkonium

*E-Mail address: wzhang@phys.sinica.edu.tw

bound state equation. We find that the effective coupling constant determined from the quarkonium bound state equation can be arbitrarily small so that the weak-coupling treatment to heavy hadron bound states in light-front QCD is explicitly achieved. Finally, the scale dependence of the effective coupling constant is analytically calculated and the similarity renormalization group β function is determined, from which the running coupling constant in small momentum transfer is given qualitatively by $\overline{\alpha}(Q^2) \sim \frac{\Lambda^2_{QCD}}{Q^2}$. Such a running coupling constant is the basic assumption in the successful Richardson $Q\overline{Q}$ potential that ensures the existence of a linear confining potential at large distance, but now can be obtained from light-front QCD.

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I. INTRODUCTION

There are two fundamental problems in QCD for hadronic physics, the quark confinement and the spontaneous breaking of chiral symmetry. These two problems are the basis for solving the low-energy hadronic bound states from QCD but none of them has been completely understood. Recently, Wilson and his collaborators proposed an approach to determine the low-energy bound states in nonperturbative QCD as a weak-coupling problem [1]. The key to eliminating necessarily nonperturbative effects is to construct a low-energy QCD Hamiltonian in which quarks and gluons have nonzero constituent masses rather than the zero masses of the current picture. The use of constituent masses cuts off the growth of the running coupling constant and makes it conceivable that the running coupling never leaves the perturbative domain. The weak-coupling approach potentially reconciles the simplicity of the constituent quark model with the complexities of QCD. The penalty for achieving this weak-coupling picture is the necessity of formulating the problem in light-front coordinates and of dealing with the complexities of renormalization. To handle the complexities of light-front renormalization, a new renormalization approach, so-called the similarity renormalization group scheme, has also recently been developed [1,2].

Based on the idea of the light-front similarity renormalization group scheme and the concept of coupling coherence [3], Perry has shown that upon a calculation to the second order, there exists a logarithmic confining potential in the resulting light-front QCD effective Hamiltonian [4]. This is a crucial finding for a practical realization of the weak-coupling treatment to nonperturbative QCD. However, the general strategy of solving hadrons through the weak-coupling treatment scheme is far to be completed. In this paper, we shall use the similarity renormalization group approach to derive a low energy heavy quark QCD Hamiltonian, based on the light-front heavy quark effective theory we developed recently [5,6]. The resulting Hamiltonian exhibits explicitly the confining and Coulomb interactions. We thereby adopt the idea of the weak-coupling approach to study the strongly interacting heavy hadronic structure on the light-front. From this investigation we explicitly provide a

realization of the weak-coupling treatment to nonperturbative QCD.

The reason of choosing heavy hadron systems as a starting example is clear: For light quark systems, both quark confinement and spontaneous chiral symmetry breaking play an essential role to the quark dynamics in hadrons. However, the success of the chiral symmetry description of the low energy hadron physics naturally indicates a chiral symmetry breaking scale ($\Lambda_{\chi SB} \sim 1\ GeV$) which is relatively larger than the confinement scale ($\Lambda_{QCD} \sim 200\ MeV$). In other words, for light quark systems, it seems to be necessary to understand the underlying mechanism of chiral symmetry breaking before we can further explore the mechanism of quark confinement. Of course, for the best description, both problems should be solved simultaneously in the same picture, but at the moment, this will certainly complicate the study on confinement. It would be nice if we could separately deal with these two most difficult but fundamental problems in QCD. For heavy quark systems, chiral symmetry is explicitly broken so that confinement is the sole nontrivial feature influencing heavy quark dynamics.

Confinement interaction may not be very sensitive but it is important in describing the QCD dynamics of quarkonium spectroscopy and their decay processes. And it should play a more important role in heavy-light quark systems, such as B and D mesons. One may argue that the mass scales for heavy and light quark systems are different. The heavy quark energy cannot run down to the usual hadronic scale of light quark systems due to heavy quark mass. Meanwhile, confining interactions must be energy scale dependent. Apparently, confinement for heavy quark systems could be very different from light quark systems. However, despite heavy and light quarks, confinement arises only from low energy gluon interactions. In other words, the confinement mechanism should be the same for both heavy and light quark systems. We thus choose heavy hadron systems without any loss of generality. In order to avoid the possible confusion about the different mass scales and to correctly extract the confining interactions in low energy heavy quark dynamics, it is convenient to work with heavy quark effective theory (HQET). The HQET is a theory of QCD in $1/m_Q$ expansion [7], where m_Q is the heavy quark mass. In HQET, the low-energy dynamics is determined

through the interacting gluons and heavy quarks by exchanging a small residual momentum of heavy quarks, which is of order Λ_{QCD}. As a result, within HQET we can indeed explore the low energy QCD dynamics for heavy quark systems in the same scale as that for light quark systems. Meanwhile, the extension of the study to light quark systems becomes obviously straightforward, although undoubtedly the corresponding result must be very complicated due to the spin dependence of the low energy interacting Hamiltonian. The spin dependent interactions on the light-front are essentially related to the chiral symmetry breaking. These spin dependent interactions in HQET are suppressed in the leading order approximation because they are $1/m_Q$ corrections and can be treated perturbatively with respect to the heavy hadron states. This is why for heavy quark systems the chiral symmetry breaking can be treated separately from the confinement.

In fact, the model-based theoretical investigations on heavy quarkonia lasted for one and half decades is recently replacing by first-principles exploration on QCD. The lattice QCD simulation may give an acceptable description for heavy quarkonium spectroscopy with manageable control over all the systematic errors [8]. The development of nonrelativistic QCD provides a general factorization formula to quarkonium annihilation and production processes so that a rigorous QCD analysis may become possible [9]. Meanwhile, in the past five years considerable progress has been made for heavy hadrons with one heavy quark, due mainly to the discovery of the so-called heavy quark symmetry (HQS) [10] and the development of the heavy quark effective theory (HQET) [7] from QCD. The HQS and HQET have in certain contents put the description of heavy hadron physics on a QCD-related and model-independent basis. Yet, a truly first-principles QCD understanding of heavy hadrons is still lacking since no good nonperturbative QCD approach is available for a direct computation of heavy hadron wavefunctions. On the other hand, in the last decade, the investigations of the light-front field theory on nonperturbative bound state problems have made some progress [11] but no real hadronic problem has been solved from which. Starting with heavy hadrons may provide a possible explicit solution of hadronic bound states in light-front QCD.

Simply speaking, the approach to achieve the QCD description of hadronic bound states that we shall study in this paper can be summarized as follows: Applying the similarity renormalization group approach to light-front QCD, we can obtain an low energy QCD Hamiltonian which is an expansion in terms of the QCD coupling constant. Then we attempt to solve from this low energy QCD Hamiltonian the strong interacting bound states as a weak-coupling problem. The weak-coupling treatment contains the following steps: (i) Compute from the similarity renormalization group scheme the low energy Hamiltonian H_λ at the low-energy cutoff λ up to the second order in coupling constant. Then separate the Hamiltonian into a nonperturbative part, $H_{\lambda 0}$ which contains not only the free Hamiltonians of quarks and gluons but also the dominant two-body interactions, and the remaining part plus the higher order contributions generated in the similarity renormalization as a perturbative term, $H_{\lambda I}$. (ii) Introduce a constituent picture which is an important step in the realization of the weak-coupling treatment of nonperturbative QCD. The constituent quarks and gluons have masses of a few hundreds MeV, and these masses are functions of the cutoff λ that must vanish when the effective theory goes back to the full QCD theory. (iii) Solve hadronic bound states with $H_{\lambda 0}$ nonperturbatively in the constituent picture and determine the cutoff dependence of the constituent masses and the coupling constant. The coupling constant g now becomes an effective one, g_λ. In the nonperturbative study of $H_{\lambda 0}$, if we could show that with a suitable choice of the low energy cutoff λ, the effective coupling constant g_λ is arbitrarily small, then a weak-coupling treatment could be applied to the low energy QCD H_λ such that the corrections from $H_{\lambda I}$ can really be computed perturbatively. (iv) There should be a limit $g_\lambda \to g_s$, where g_s is the fixed physical coupling constant measured at the hadronic mass scale, such that all the constituent quarks and gluons become current ones again. Then the effective low energy theory returns back to the full QCD theory. If everything listed above works well, we arrive at a true weak-coupling QCD theory of the strong interaction for hadrons.

In the following, I begin with a general bare interacting Hamiltonian with a high energy cutoff Λ that removes the usual ultraviolet (UV) divergences. Then using the light-front

similarity renormalization group, we construct a Hamiltonian under the low energy cutoff λ. The low energy cutoff is introduced via a smearing function in the similarity renormalization group that effectively integrates over all the modes above the cutoff λ. The choice of the smearing function in this paper is much simpler in comparison to the original setup [1]. Applying this general formula to the light-front heavy quark effective theory (light-front HQET) we developed recently [5,6], a low energy confining QCD Hamiltonian can be explicitly obtained for heavy hadron systems. Consequently, a clear light-front picture of quark confinement emerges. Furthermore, based on the idea of weak-coupling approach, we use this confining Hamiltonian to study the light-front heavy hadron bound states nonperturbatively, and we can then provide an explicit description of the weak-coupling treatment in the light-front HQET for heavy hadrons.

The present paper is organized as follows. In Section 2, we present a general procedure of constructing a low energy light-front QCD Hamiltonian in the similarity renormalization group scheme and discuss the possible existence of confining interaction in such a derivation. In Section 3, applying the general procedure to the light-front HQET of QCD, we further derive the heavy quark confining Hamiltonian at the low-energy scale. In Section 4, the light-front heavy hadronic bound state equations in the constituent picture are developed within the scheme of the weak-coupling treatment. A light-front picture of quark confinement for heavy hadrons is illustrated in Section 5. In Section 6, the heavy hadron bound state equation is solved for quarkonia with a Gaussian-type light-front wavefunction ansatz, from which the scale dependence of effective coupling constant is determined as a solution of the similarity renormalization group equation on the quarkonium binding energy in Section 7. The low energy running coupling constant is also qualitatively obtained. In Section 8, a connection between the low energy effective theory to the full QCD theory is explored and the consistency is provided. Finally, a summary is presented in Section 9.

II. LOW ENERGY QCD HAMILTONIAN IN SIMILARITY RENORMALIZATION GROUP SCHEME

We begin with the general formulation of the similarity renormalization group approach to construct a low energy QCD Hamiltonian. In general, for a given bare Hamiltonian, $H^B = H_0^B + H_I^B$, where H_0^B is a bare free Hamiltonian and E_i is assumed to be its eigenvalue, the similarity renormalization group approach leads to the following Hamiltonian at the low energy cutoff λ (for a detailed derivation, see Ref. [1]):

$$\begin{aligned} H_\lambda &= \left(H_{0\lambda}^B + H_{I\lambda}^B\right) + \left(\underbrace{[H_{I\lambda'}^B, H_{I\lambda' T}^B]}_{R}\right) \\ &\quad + \left(\underbrace{[\underbrace{[H_{I\lambda''}^B, H_{I\lambda'' T}^B]}_{R'}, H_{I\lambda' T}^B]}_{R} + \underbrace{[H_{I\lambda'}^B, \underbrace{[H_{I\lambda''}^B, H_{I\lambda'' T}^B]}_{T'}]}_{R}\right) + \ldots \\ &= H_\lambda^{(0)} + H_\lambda^{(2)} + H_\lambda^{(3)} + \ldots, \end{aligned} \tag{2.1}$$

where $H_{\lambda ij}^B = f_{\lambda ij} H_{ij}^B$ (we use the notation $A_{ij} = \langle i|A|j\rangle$), $H_{I\lambda T ij}^B = -\frac{1}{E_j - E_i}\left(\frac{d}{d\lambda} f_{\lambda ij}\right) H_{Iij}^B$, and

$$\underbrace{X_{\lambda' ij}}_{R} = -f_{\lambda ij} \int_\lambda^\infty d\lambda' X_{\lambda' ij}, \tag{2.2}$$

$$\underbrace{X_{\lambda' ij}}_{T} = -\frac{1}{E_j - E_i}\left(\frac{d}{d\lambda} f_{\lambda ij}\right) \int_\lambda^\infty d\lambda' X_{\lambda' ij} + \frac{1}{E_j - E_i}(1 - f_{\lambda ij}) X_{\lambda ij}. \tag{2.3}$$

The function $f_{\lambda ij} = f(x_{\lambda ij})$ is a smearing function in the similarity renormalization group, and $x_{\lambda ij} = \frac{E_j - E_i}{E_i + E_j + \lambda}$. The smearing function is introduced to force the Hamiltonian H_λ becoming a band diagonal form in energy space. This requires the following properties for $f_{\lambda ij}$: when $x < 1/3$, $f = 1$; when $x > 2/3$, $f = 0$; and f may be a smooth function from 1 to 0 for $1/3 \leq x \leq 2/3$. Thus, through the similarity renormalization group, we eliminate the interactions between the states well-separated in energy and generate the Hamiltonian of eq.(2.1). The expansion of eq.(2.1) in terms of the interaction coupling constant brings in order by order the full theory corrections to this band diagonal low energy Hamiltonian.

Explicitly, the bare Hamiltonian H^B input in the above formulation can be obtained from the canonical Lagrangian with a high energy cutoff that removes the usual UV divergences.

For light-front QCD dynamics, the bare Hamiltonian in our consideration is the canonical light-front QCD Hamiltonian that can be either obtained from the canonical procedure in the light-front gauge [12,13] or generated from the light-front power counting rules [1]. Instead of the cutoff on the field operators which is introduced in ref. [1], we shall use in this paper a vertex cutoff to every vertex in the bare Hamiltonian:

$$\theta(\Lambda^2/P^+ - |p_i^- - p_f^-|), \tag{2.4}$$

where p_i^- and p_j^- are the initial and final state light-front energies respectively between the vertex, Λ is the UV cutoff parameter, and P^+ the total light-front longitudinal momentum of the system we are interested in. Eq.(2.4) is also called the local cutoff in light-front perturbative QCD [19]. All the Λ-dependences in the final bare Hamiltonian are removed by the counterterms so that the bare Hamiltonian H^B used in eq.(2.1) has already been renormalized as $\Lambda \to \infty$. The use of eq.(2.4) largely simplifies the analysis on the cutoff scheme in ref. [1].

Meanwhile, in similarity renormalization group calculation, we should also give an explicit form of the smearing function $f_{\lambda ij}$. One of the simplest smearing functions that satisfies the requirements of the similarity renormalization group scheme is a theta-function:

$$f_{\lambda ij} = \theta(\frac{1}{2} - x_{\lambda ij}). \tag{2.5}$$

However, using the definition of $x_{\lambda ij}$, we can further replace the above smearing function by the following form on the light-front:

$$f_{\lambda ij} = \theta(\frac{\lambda^2}{P^+} - |\Delta P_{ij}^-|), \tag{2.6}$$

where λ is a low energy cutoff, and $\Delta P_{ij}^- = P_i^- - P_j^-$ is the light-front free energy difference between the initial and final states of the physical processes. The light-front free energies of the initial and final states are defined as sums over the light-front free energies of the constituents in the states. The smearing function eq.(2.6) satisfies the requirements for the similarity renormalization group approach although it is not a smooth function.

Throughout this paper, we shall always use the definition of eq.(2.6). Thus, the Hamiltonian (2.1) can be reduced to

$$H_{\lambda ij} = \theta(\lambda^2/P^+ - |\Delta P^-_{ij}|)\left\{H^B_{ij} + \sum_k H^B_{Iik}H^B_{Ikj}\Big[\frac{g_{\lambda jik}}{\Delta P^-_{ik}} + \frac{g_{\lambda ijk}}{\Delta P^-_{jk}}\Big] + \cdots\right\}. \tag{2.7}$$

The front factor (the theta-function) in the above equation indicates that the Hamiltonian H_λ describes the low energy interactions (with respect to the cutoff λ). Therefore, λ should be a value of the hadronic mass scale. Eq.(2.7) shows that the low energy Hamiltonian is apparently a modified Hamiltonian perturbative expansion. The function $g_{\lambda ij}$ in eq.(2.7) is

$$g_{\lambda ijk} = \int_{\lambda^2/P^+}^{\infty} d(\lambda^2/P^+) f_{\lambda' ik}\frac{d}{d(\lambda'^2/P^+)}f_{\lambda' jk} = \theta(|\Delta P^-_{jk}| - \lambda^2/P^+)\theta(|\Delta P^-_{jk}| - |\Delta P^-_{ik}|). \tag{2.8}$$

The theta function in eq.(2.8) guarantees that the singularity coming from the small energy denominators in the usual Hamiltonian perturbation theory does not occur in the above formulation.

Up to this point, we have introduced two cutoffs, Λ and λ. The cutoff Λ is used to remove the UV divergences as in the usual perturbation theory. While the low energy cutoff λ is introduced to separate the high and low energy state interactions, it is indeed a scale parameter described by the light-front similarity renormalization group. The similarity renormalization group transformations integrate out all physical degrees of freedom above the cutoff λ and generate the low energy Hamiltonian H_λ. Formally, when $\lambda \to \infty$, $f_{\lambda ij} = 1$, and eqs.(2.2) and (2.3) vanish so that $H_\lambda \to H^B$. In practice, we shall take $\lambda \sim 1$ GeV. Of course, the final physical observables must be λ independent. As a consequence of this condition, the coupling constant and the constituent masses in the Hamiltonian H_λ become functions of this low energy cutoff λ. It is these cutoff (or scale) dependences that link the effective theory to the full theory in the limit $g \to g_s$.

Next, to calculate explicitly the low energy effective QCD Hamiltonian, we begin with the canonical QCD theory. It is convenient to use the light-front two-component formulation of canonical QCD in the light-front gauge $A^+_a = 0$. The bare light-front QCD Hamiltonian in such a two-component formalism is given by [13]

$$H^B = \int_c dx^+ d^2x_\perp \Big(\mathcal{H}_0 + \mathcal{H}_I\Big). \tag{2.9}$$

Here $\int_c$ means that the local cutoff, eq.(2.4), has been imposed, and

$$\mathcal{H}_0 = \frac{1}{2}(\partial^i A_a^j)(\partial^i A_a^j) + \xi^\dagger \frac{-\partial_\perp^2 + m^2}{i\partial^+}\xi, \tag{2.10}$$

$$\mathcal{H}_I = \mathcal{H}_{qqg} + \mathcal{H}_{ggg} + \mathcal{H}_{qqqq} + \mathcal{H}_{qqgg} + \mathcal{H}_{gggg} + \text{counterterms}, \tag{2.11}$$

with

$$\begin{aligned}\mathcal{H}_{qqg} = g\xi^\dagger \Big\{ &-2\left(\frac{1}{\partial^+}\right)(\partial\cdot A_\perp) + \tilde\sigma\cdot A_\perp \left(\frac{1}{\partial^+}\right)(\tilde\sigma\cdot\partial_\perp + m) \\ &+ \left(\frac{1}{\partial^+}\right)(\tilde\sigma\cdot\partial_\perp - m)\tilde\sigma\cdot A_\perp\Big\}\xi,\end{aligned} \tag{2.12}$$

$$\mathcal{H}_{ggg} = g f^{abc}\left\{\partial^i A_a^j A_b^i A_c^j + (\partial^i A_a^i)\left(\frac{1}{\partial^+}\right)(A_b^j \partial^+ A_c^j)\right\}, \tag{2.13}$$

$$\begin{aligned}\mathcal{H}_{qqgg} = g^2 \Big\{ &\xi^\dagger \tilde\sigma\cdot A_\perp \left(\frac{1}{i\partial^+}\right)\tilde\sigma\cdot A_\perp \xi \\ &+ 2\left(\frac{1}{\partial^+}\right)(f^{abc} A_b^i \partial^+ A_c^i)\left(\frac{1}{\partial^+}\right)(\xi^\dagger T^a \xi)\Big\},\end{aligned} \tag{2.14}$$

$$\mathcal{H}_{qqqq} = 2g^2\left\{\left(\frac{1}{\partial^+}\right)(\xi^\dagger T^a\xi)\left(\frac{1}{\partial^+}\right)(\xi^\dagger T^a\xi)\right\}, \tag{2.15}$$

$$\begin{aligned}\mathcal{H}_{gggg} = \frac{g^2}{4} f^{abc} f^{ade}\Big\{ &A_b^i A_c^j A_d^i A_e^j \\ &+2\left(\frac{1}{\partial^+}\right)(A_b^i\partial^+ A_c^i)\left(\frac{1}{\partial^+}\right)(A_d^j\partial^+ A_e^j)\Big\},\end{aligned} \tag{2.16}$$

where $A_\perp = T^a A_{a\perp}$ is the transverse component of the gauge field, T^a is the generator of SU(3) color group, and ξ is the two-component form of the light-front quark field:

$$\psi_+ = \Lambda^+\psi = \begin{bmatrix}\xi\\ 0\end{bmatrix} \quad , \quad \psi_- = \Lambda^-\psi = \begin{bmatrix} 0 \\ \left(\frac{1}{i\partial^+}\right)[\tilde\sigma^i(i\partial^i + gA^i) + im]\xi\end{bmatrix}. \tag{2.17}$$

The notation $\tilde\sigma$ is defined by: $\tilde\sigma^1 = \sigma^2, \tilde\sigma^2 = -\sigma^1$ (the Pauli matrices). This comes from the use of the light-front γ-representation:

$$\begin{aligned}\gamma^0 &= \begin{bmatrix} 0 & -i \\ i & 0\end{bmatrix} \quad , \quad \gamma^3 = \begin{bmatrix} 0 & i \\ i & 0\end{bmatrix}, \\ \gamma^1 &= \begin{bmatrix} -i\sigma^2 & 0 \\ 0 & i\sigma^2\end{bmatrix} \quad , \quad \gamma^2 = \begin{bmatrix} i\sigma^1 & 0 \\ 0 & -i\sigma^1\end{bmatrix},\end{aligned} \tag{2.18}$$

which is different from the one we used in ref. [13]. The reason of using this new γ-representation is that it leads to not only a realization of the two-component form for the light-front fermion field, but also a correct correspondence of the fermion spin operator $S^i \sim \sigma^i$ on the light-front. Counterterms are added to eq.(2.11) in order to remove all the Λ-dependence. Thus the coupling constant g in eqs.(2.12–2.16) is a perturbatively renormalized coupling constant.

Upon the calculation to the second order in the coupling constant, we obtain an effective Hamiltonian (2.7) in $q\overline{q}$ sector (with the initial and final states, $|i\rangle = b^\dagger(p_1,\lambda_1)d^\dagger(p_2,\lambda_2)|0\rangle$ and $|j\rangle = b^\dagger(p_3,\lambda_3)d^\dagger(p_4,\lambda_4)|0\rangle$, respectively, where p_i and λ_i denote the respective momentum and helicity of a quark on the light-front):

$$
\begin{aligned}
H_{\lambda ij} = \theta(\lambda^2/P^+ - |\Delta P^-_{ij}|)\Bigg\{ & \langle j| : H^B : |i\rangle - \frac{g^2}{2\pi^2}\lambda^2 C_f \frac{1}{P^+}\ln\epsilon + \text{mass counterterms} \\
& - g^2(T^a)(T^a)\frac{\theta(q^+)}{q^+}\chi^\dagger_{\lambda_1}\left[2\frac{q^{i'}_\perp}{q^+} - \frac{\tilde\sigma\cdot p_{3\perp} - im}{[p^+_3]}\tilde\sigma^{i'} - \tilde\sigma^{i'}\frac{\tilde\sigma\cdot p_{1\perp}+im}{[p^+_1]}\right]\chi_{\lambda_3} \\
& \times \chi^\dagger_{-\lambda_2}\left[2\frac{q^{i'}_\perp}{q^+} - \frac{\tilde\sigma\cdot p_{2\perp}+im}{[p^+_2]}\tilde\sigma^{i'} - \tilde\sigma^{i'}\frac{\tilde\sigma\cdot p_{4\perp}-im}{[p^+_4]}\right]\chi_{-\lambda_4}F_{rij}\Bigg\}. \quad (2.19)
\end{aligned}
$$

Here $P^+ = p^+_1 + p^+_2 = p^+_3 + p^+_4$, $\Delta P^-_{ij} = p^-_1 + p^-_2 - p^-_3 - p^-_4$, $: H^B :$ represents a normal ordering, where the instantaneous interaction contribution to the quark self-energy has been included in the self-energy calculation which is given by the mass counterterms and the logarithmic divergence in (2.19), the color factor $C_f = (T^aT^a) = (N^2-1)/2N$, $N = 3$ the total numbers of colors, and ϵ is an infrared longitudinal momentum cutoff. Since $\ln\epsilon$ is an infrared divergence, it cannot be removed by mass counterterms. In gauge symmetry, this divergence must be canceled in the physical sector (and this is true as we will see later). The last term in (2.19) is the one-gluon exchange contribution to the low energy Hamiltonian. The momentum q is carried by the exchange gluon: $q^+ = p^+_1 - p^+_3 = p^+_4 - p^+_2$, $q_\perp = p_{1\perp} - p_{3\perp} = p_{4\perp} - p_{2\perp}$, χ_{λ_i} denotes a helicity eigenstate. The factor F_{rij} arises from the similarity renormalization group scheme:

$$
F_{rij} = \Bigg\{\theta(\Lambda^2/P^+ - |p^-_1 - p^-_3 - q^-|)\theta(\Lambda^2/P^+ - |p^-_4 - p^-_2 - q^-|)
$$

$$\times \left[\frac{\theta(|p_1^- - p_3^- - q^-| - \frac{\lambda^2}{P^+})\theta(|p_1^- - p_3^- - q^-| - |p_4^- - p_2^- - q^-|)}{p_1^- - p_3^- - q^-} \right.$$
$$\left. + \frac{\theta(|p_4^- - p_2^- - q^-| - \frac{\lambda^2}{P^+})\theta(|p_4^- - p_2^- - q^-| - |p_1^- - p_3^- - q^-|)}{p_4^- - p_2^- - q^-} \right]$$
$$\left. + (p_1 \longleftrightarrow p_3 \ , \quad p_2 \longleftrightarrow p_4 \ , \quad q \longrightarrow -q) \right\}. \tag{2.20}$$

All light-front energies in eq.(2.20) are on mass-shell: $p_i^- = \frac{p_{i\perp}^2 + m_i^2}{p_i^+}$ and $q^- = \frac{q_\perp^2}{q^+}$.

If we continue to evaluate all the terms in the expansion of eq.(2.1), the resulting Hamiltonian is the exact QCD Hamiltonian. In practice, we should only consider the leading and the next-to-leading terms, i.e., eq.(2.19), as a starting effective Hamiltonian. The basic idea to realize a weak-coupling treatment of QCD for hadrons is whether we can solve hadron states from this effective Hamiltonian (2.19) with an arbitrary small coupling constant g such that the higher order corrections in (2.1) can be handled perturbatively. From the success of the constituent quark model, we understand that a necessity for such a realization is the existence of a confining interaction in (2.19).

Naively, we know that any weak-coupling Hamiltonian derived from QCD will have only Coulomb-like interactions, and confinement can only be exhibited in a strong-coupling theory. However, Perry has recently found that upon to the second order calculation of the low energy Hamiltonian, a logarithmic confining potential has already occurred [4]. Explicitly, the two-body quark-antiquark interaction from the first term of eq.(2.19) is the instantaneous gluon exchange interaction $\mathcal{H}_{qqqq}$ which has the form in the momentum space:

$$-\frac{1}{(q^+)^2}. \tag{2.21}$$

The confinement must be associated with the interaction where $q^+ \to 0$. This is because only these particles with zero longitudinal momentum can occupy the light-front vacuum state [1]. Thus, we need only to analyze the feature of the effective Hamiltonian when $q^+ \to 0$. For $q^+ \to 0$, the dominant second order contribution from one transverse gluon exchange interaction in eq.(2.19) is given by

$$\left[\frac{1}{(q^+)^2} \frac{q_\perp^2}{q_\perp^2} + O(\frac{1}{q^+}) \right] \theta(q^- - \lambda^2/P^+). \tag{2.22}$$

In the usual perturbative calculation, no such a θ-function is attached in the above equation. Thus these two dominant contributions from the instantaneous and one-gluon exchange interactions are exactly cancelled when $q^+ \to 0$. Only Coulomb-type interaction (the terms $\sim O(\frac{1}{q^+})$) remains. However, in the light-front similarity renormalization group scheme, the one-gluon exchange contribution to eq.(2.22) only contains these gluons with energy being greater than the energy cutoff λ^2/P^+. As a result, the instantaneous gluon exchange term $1/(q^+)^2$ remains uncancelled if the gluon energy, $q_\perp^2/q^+$, is less than λ^2/P^+. The remaining uncancelled instantaneous interaction contains an infrared divergence and a finite part contribution (for a detailed derivation, see section V). The divergence part is cancelled precisely for physical states by the same divergence in the quark self-energy correction [see (2.19)]. The remaining finite part corresponds to a logarithmic confining potential:

$$b_\lambda \ln|x^-| + c_\lambda \ln\Big(\frac{\lambda^2|x_\perp|^2}{P^+}\Big).$$

The above result is first obtained by Perry with the use of the concept of coupling coherence and a slightly different renormalization scheme [4] (also see a oversimple derivation given by Wilson [15]). Here the derivation is purely based on the light-front similarity renormalization scheme [1].

One may argue that the existence of such a confining potential in H_λ may only be an artificial effect designed in the renormalization scheme we used. If we included the interaction with the exchange gluon energy below the cutoff, then the instantaneous interaction would be completely cancelled, and no such confining potential should exist, as expected in the usual perturbation computation. Wilson has pointed out that the set up of the new renormalization scheme is motivated by the idea that the gluon mass must be nonzero in the low energy domain (a constituent picture), which could be regarded as an effect of the nontrivial low energy gluon interactions. The cutoff λ is of the same order as the constituent gluon mass. Thus the gluon energy cannot run down below the cutoff λ. The existence of the confining potential is a result of the low-energy gluon interactions. In contrast, the photon mass in QED is zero at any scale. The instantaneous photon exchange interaction

is always cancelled by the corresponding one transverse photon exchange interaction, and only the Coulomb interaction is left. Thus, the above confining potential can only exist in QCD [15]. More detailed discussions will be seen in the subsequent sections.

Yet, even up to the second order, the effective QCD Hamiltonian H_λ is already very complicated. In order to to examine the above ideas of confining mechanism and to develop explicitly a weak-coupling treatment approach of nonperturbative QCD to hadronic bound states, in the next section we shall utilize the above formulation to heavy quark systems. We find that the low-energy QCD Hamiltonian H_λ for heavy quarks can be largely simplified and an analytic form consisting of the confining potential and Coulomb potential emerges.

III. LOW-ENERGY HEAVY QUARK CONFINING HAMILTONIAN

In the past few years, QCD has been made a numerous progresses in understanding the heavy hadron structure, due mainly to the discovery of heavy quark symmetry by Isgur and Wise [10], and the development of the heavy quark effective theory by Georgi et al. [7]. Very recently, we have reformulated the heavy quark effective theory from QCD on the light-front [5,6], which may provide a convenient basis for the further study of the nonperturbative structures of heavy hadron. Now we use the light-front HQET to derive the low energy heavy quark effective Hamiltonian in the similarity renormalization group scheme.

A. Light-front heavy quark effective theory

The light-front heavy quark effective Lagrangian derived from QCD lagrangian $\mathcal{L} = \overline{Q}(i\not{D} - m_Q)Q$ as a $1/m_Q$ expansion is given in refs. [5,6]:

$$\mathcal{L} = \frac{2}{v^+}\mathcal{Q}^\dagger_{v+}(iv\cdot D)\mathcal{Q}_{v+} - \sum_{n=1}^{\infty}\Big(\frac{1}{m_Q v^+}\Big)^n \mathcal{Q}^\dagger_{v+}\Big\{(i\vec{\alpha}\cdot\vec{D})(-iD^+)^{n-1}(i\vec{\alpha}\cdot\vec{D})\Big\}\mathcal{Q}_{v+}, \tag{3.1}$$

where $\mathcal{Q}_{v+}$ is the light-front dynamical component of the heavy quark field after the phase redefinition:

$$Q(x) = e^{-im_Q v\cdot x}(\mathcal{Q}_{v+}(x) + \mathcal{Q}_{v-}(x)), \tag{3.2}$$

v^μ the four velocity of the heavy hadrons, $P^\mu = M_H v^\mu$ with $v^2 = 1$ and M_H being the heavy hadron mass, $\vec{\alpha}\cdot\vec{D} \equiv \alpha_\perp \cdot D_\perp - \frac{1}{v^+}(\alpha_\perp \cdot v_\perp + \beta)D^+$, and D^μ is the usual covariant derivative. The corresponding light-front heavy quark bare Hamiltonian density is given by

$$\mathcal{H} = \frac{1}{iv^+}\mathcal{Q}^\dagger_{v+}(v^-\partial^+ - 2v_\perp\cdot\partial_\perp)\mathcal{Q}_{v+} - \frac{g}{v^+}\mathcal{Q}^\dagger_{v+}(v\cdot A)\mathcal{Q}_{v+} + \sum_{n=1}^{\infty}\Big(\frac{1}{m_Q v^+}\Big)^n \mathcal{Q}^\dagger_{v+}\Big\{(i\vec{\alpha}\cdot\vec{D})(-iD^+)^{n-1}(i\vec{\alpha}\cdot\vec{D})\Big\}\mathcal{Q}_{v+}. \tag{3.3}$$

In the large m_Q limit, only the leading (spin and mass independent) Hamiltonian is remained. In other words, the phase redefinition (3.2) removes the dominant piece of the space-time dependence of the heavy quark. The remaining dependence is only due to the residual momentum of the heavy quark in the heavy hadrons. The $1/m_Q^n$ terms ($n \geq 1$) in (3.3) can be regarded as perturbative corrections to the leading order operators and states. Therefore, the heavy quark mass m_Q is indeed a factorization scale for separating heavy quark short and long distance dynamics. To determine confining interactions between two heavy quarks or a heavy quark with a light quark, only the leading heavy quark Hamiltonian plays an essential role. We choose the light-front gauge $A^+ = 0$, the leading-order bare QCD Hamiltonian density (corresponding to the limit of $m_Q \to \infty$) is given from (3.3):

$$\begin{aligned}\mathcal{H}_{ld} &= \frac{1}{iv^+}\mathcal{Q}^\dagger_{v+}(v^-\partial^+ - 2v_\perp\cdot\partial_\perp)\mathcal{Q}_{v+} \\ &\quad - \frac{2g}{v^+}\mathcal{Q}^\dagger_{v+}\Big\{v^+\Big[\Big(\frac{1}{\partial^+}\Big)\partial_\perp\cdot A_\perp\Big] - v_\perp\cdot A_\perp\Big\}\mathcal{Q}_{v^+} \\ &\quad + 2g^2\Big(\frac{1}{\partial^+}\Big)\Big(\mathcal{Q}^\dagger_{v+}T^a\mathcal{Q}_{v^+}\Big)\Big(\frac{1}{\partial^+}\Big)\Big(\mathcal{Q}^\dagger_{v+}T^a\mathcal{Q}_{v^+}\Big) \\ &= \mathcal{H}_0 + \mathcal{H}_{qqg} + \mathcal{H}_{qqqq}.\end{aligned} \tag{3.4}$$

Note that besides the leading term in eq.(3.3), the above bare Hamiltonian has already also included the relevant terms from the gauge field part, $-\frac{1}{2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})$, of the QCD Lagrangian. These terms come from the elimination of the unphysical gauge degrees of freedom, the longitudinal component A^-_a, in the light-front gauge (see a detailed derivation in refs. [12,13]). Eq(3.4) has obviously the spin and flavour heavy quark symmetry, or simply the heavy quark symmetry.

In momentum space, the free part of the bare light-front effective heavy quark Hamiltonian can be simply expressed by

$$H_0 = \sum_\lambda \int [d^3\bar{k}] \frac{1}{v^+}(2v_\perp \cdot k_\perp - v^- k^+)\Big\{b_v^\dagger(k,\lambda) b_v(k,\lambda) + d_v^\dagger(k,\lambda) d_v(k,\lambda)\Big\}$$
$$= \sum_\lambda \int [d^3\bar{k}] k^- \Big\{b_v^\dagger(k,\lambda) b_v(k,\lambda) + d_v^\dagger(k,\lambda) d_v(k,\lambda)\Big\}, \tag{3.5}$$

where k is the residual momentum of heavy quarks, $p^\lambda = m_Q v^\lambda + k^\lambda$, and λ its helicity. We have introduced the notation for the space components of light-front momentum $(p^+, p_\perp) \equiv \bar{p}$ so that $[d^3\bar{p}] \equiv \frac{dp^+ d^2p_\perp}{2(2\pi)^3}$. The operator $b_v^\dagger(k,\lambda)$ $[d_v^\dagger(k,\lambda)]$ creates a heavy quark [antiquark] with velocity v, residual momentum k and helicity λ,

$$\{b_v(k,\lambda),\ b_{v'}^\dagger(k',\lambda')\} = \{d_v(k,\lambda),\ d_{v'}^\dagger(k',\lambda')\} = 2(2\pi)^3 \delta_{vv'}\delta_{\lambda\lambda'}\delta^3(\bar{k} - \bar{k}'), \tag{3.6}$$

where $\delta^3(\bar{k} - \bar{k}') \equiv \delta(k^+ - k'^+)\delta^2(k_\perp - k'_\perp)$.

Eq.(3.5) means that after the redefinition of the heavy quark field, the heavy quark in the light-front HQET carries the effective free light-front energy

$$k^- = \frac{1}{v^+}(2v_\perp \cdot k_\perp - v^- k^+). \tag{3.7}$$

The meaning of this result becomes more transparent if we expand the light-front energy dispersion relation of the heavy quarks as an inverse power of m_Q:

$$p^- = \frac{p_\perp^2 + m_Q^2}{p^+} = m_Q v^- + \frac{1}{v^+}(2v_\perp \cdot k_\perp - v^- k^+) + O(1/m_Q)$$
$$\overset{m_Q \to \infty}{=} m_Q v^- + k^-. \tag{3.8}$$

We see that the mass part $(m_Q v^-)$ has been removed by the phase redefinition of (3.2). Thus, eq.(3.4) describes effectively the "lighten" heavy quark dynamics with respect to its residual momentum. In other words, the heavy quarks in the effective theory have the same energy scaling behavior as the light ones.

The above leading Hamiltonian (or Lagrangian) is the basis of the QCD-based description for heavy hadrons containing a single heavy quark, such as B and D mesons. However, as recently pointed out by Mannel et al. [16,17] the purely heavy quark leading Lagrangian

may be not appropriate for the description of heavy quarkonia states. This is because the anomalous dimension of the QCD radiative correction to the $Q\overline{Q}$ currents contains an infrared singularity in the limit of two heavy constituents having equal velocity. Such an infrared singularity is a long distance effect and should be absorbed into quarkonium states. To avoid this problem, they argued that one may incorporate the effective Hamiltonian with at least the first order kinetic energy term into the leading Hamiltonian [17]. The kinetic energy in light-front HQET is given by [5]

$$\mathcal{H}_{kin} = -\frac{1}{m_Q v^+}\mathcal{Q}^\dagger_{v+}\left\{\partial^2_\perp - \frac{2v_\perp\cdot\partial_\perp}{v^+}\partial^+ + \frac{v^-}{v^+}\partial^{+2}\right\}\mathcal{Q}_{v+}. \tag{3.9}$$

As a consequence, in the heavy mass limit, quarkonia have spin symmetry but no flavour symmetry. In momentum space,

$$H_{kin} = \frac{1}{m_Q v^+}\sum_\lambda \int [d^3\bar{k}]\Big(k^2_\perp - 2v_\perp\cdot k_\perp\frac{k^+}{v^+} + \frac{v^-}{v^+}k^{+2}\Big) \times\Big\{b^\dagger_v(k,\lambda)b_v(k,\lambda) + d^\dagger_v(k,\lambda)d_v(k,\lambda)\Big\}. \tag{3.10}$$

The kinetic energy of (3.10) can be simply obtained by expanding (3.8) up to order $1/m_Q$. We will discuss later the effect of this kinetic energy in the determination of heavy quarkonium bound states.

B. Low-energy effective Hamiltonian for heavy quarkonia

Within light-front HQET, we can follow the procedure described in the previous section to find the effective low energy QCD Hamiltonian for $Q\overline{Q}$ systems. The bare Hamiltonian for $Q\overline{Q}$ systems is given by (3.4) plus the leading kinetic Hamiltonian (3.9) for both heavy quark and antiquark, where two heavy constituents have the same velocity carried by the heavy quarkonia. The kinetic energy which is of order Λ_{QCD}/m_Q, is at most the same order as the Coulomb interaction. It may affect on the quarkonium bound states but not on the derivation of the long distance quark interactions. In fact, the success of the potential-model description indicates that the scalar interactions between the two heavy constituents

are flavour-independent. In other words, the confining interaction in $Q\overline{Q}$ states should be independent of the kinetic energy (3.9). Therefore, we may treat the kinetic energy as the same as the instantaneous $Q\overline{Q}$ interaction [the last term in eq.(3.4)]. In the derivation of the low energy heavy quark Hamiltonian, the free Hamiltonian used in similarity renormalization group scheme is then simply given by eq.(3.5).

With the above consideration, it is easy to find that the leading order contribution to the low energy effective Hamiltonian is the low energy part of the heavy quark effective bare Hamiltonian,

$$\begin{aligned} H^{(0)}_{\lambda ij} = \theta(\frac{\lambda^2}{P^+} - |\Delta P^-_{ij}|)\langle j| \int dx^- d^2x_\perp \Big\{ & \frac{1}{iv^+}\mathcal{Q}^\dagger_{v+}(v^-\partial^+ - 2v_\perp\cdot\partial_\perp)\mathcal{Q}_{v+} \\ & -\frac{2g}{v^+}\mathcal{Q}^\dagger_{v+}\Big\{v^+\Big[\Big(\frac{1}{\partial^+}\Big)\partial_\perp\cdot A_\perp\Big] - v_\perp\cdot A_\perp\Big\}\mathcal{Q}_{v^+} \\ & +2g^2\Big(\frac{1}{\partial^+}\Big)\Big(\mathcal{Q}^\dagger_{v+}T^a\mathcal{Q}_{v^+}\Big)\Big(\frac{1}{\partial^+}\Big)\Big(\mathcal{Q}^\dagger_{v+}T^a\mathcal{Q}_{v^+}\Big) \\ & -\frac{1}{m_Q v^+}\mathcal{Q}^\dagger_{v+}\Big[\partial^2_\perp - \frac{2v_\perp\cdot\partial_\perp}{v^+}\partial^+ + \frac{v^-}{v^+}\partial^{+2}\Big]\mathcal{Q}_{v+}\Big\}|i\rangle, \end{aligned} \quad (3.11)$$

plus all other $1/m_Q$ terms in (3.3) as well as the light quark and gluon full QCD Hamiltonian that has not been included in the above equation [see (2.9)]. Here the initial and final states are defined by $|i\rangle = b^\dagger_v(k_1,\lambda_1)d^\dagger_v(k_2,\lambda_2)|0\rangle$ and $|j\rangle = b^\dagger_v(k_3,\lambda_3)d^\dagger_v(k_4,\lambda_4)|0\rangle$, respectively, $P^+ = p^+_1 + p^+_2 = (m_Q + m_{\overline{Q}})v^+ + k^+_1 + k^+_2 = (m_Q + m_{\overline{Q}})v^+ + k^+_1 + k^+_2$.

The next-to-leading order contribution contains two different parts, $H^{(2)}_\lambda = H^{(2)}_{\lambda 1} + H^{(2)}_{\lambda 2}$, where $H^{(2)}_{\lambda 1}$ is the self-energy correction,

$$\begin{aligned} H^{(2)}_{\lambda 1} = & \theta(\frac{\lambda^2}{P^+} - |\Delta P^-_{ij}|)[2(2\pi)^3]^2\delta^3(\bar{k}_1 - \bar{k}_3)\delta^3(\bar{k}_2 - \bar{k}_4)\delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4}(4g^2)(T^aT^a) \\ & \times\int[d^3\bar{k}]\Big\{\frac{\theta(k^+_1 - k^+)}{k^+_1 - k^+}\Big(\frac{(k_1-k)^{i'}_\perp}{k^+_1 - k^+} - \frac{v^{i'}}{v^+}\Big)^2\theta(\frac{\Lambda^2}{P^+} - |k^-_1 - k^- - (k_1-k)^-|) \\ & \qquad\times\frac{\theta(|k^-_1 - k^- - (k_1-k)^-| - \frac{\lambda^2}{P^+})}{k^-_1 - k^- - (k_1-k)^-} \\ & + \frac{\theta(k^+_2 - k^+)}{k^+_2 - k^+}\Big(\frac{(k_2-k)^{i'}_\perp}{k^+_2 - k^+} - \frac{v^{i'}}{v^+}\Big)^2\theta(\frac{\Lambda^2}{P^+} - |k^-_2 - k^- - (k_2-k)^-|) \\ & \qquad\times\frac{\theta(|k^-_2 - k^- - (k_2-k)^-| - \frac{\lambda^2}{P^+})}{k^-_2 - k^- - (k_2-k)^-}\Big\} \\ = & \theta(\frac{\lambda^2}{P^+} - |\Delta P^-_{ij}|)[2(2\pi)^3]^2\delta^3(\bar{k}_1 - \bar{k}_3)\delta^3(\bar{k}_2 - \bar{k}_4)\delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4} \end{aligned}$$

$$\times \frac{-8g^2C_f}{P^+}\int \frac{dx_1 d^2\kappa_{1\perp}}{2(2\pi)^3}\frac{\theta(x-x_1)}{(x-x_1)^2}F(x-x_1,\kappa_\perp-\kappa_{1\perp},M_H)$$
$$\times \frac{(\kappa_\perp-\kappa_{1\perp})^2}{(\kappa_\perp-\kappa_{1\perp})^2+(x-x_1)^2M_H^2}, \tag{3.12}$$

and $H_{\lambda 2}^{(2)}$ is the $Q\overline{Q}$ interaction,

$$\begin{aligned}
H_{\lambda 2}^{(2)} &= \theta(\frac{\lambda^2}{P^+}-|\Delta P_{ij}^-|)\delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4}2(2\pi)^3\delta^3(\bar{k}_1+\bar{k}_2-\bar{k}_3-\bar{k}_4)(-4g^2)(T^a)(T^a)\\
&\times\frac{1}{q^+}\left(\frac{q_\perp^{i'}}{q^+}-\frac{v^{i'}}{v^+}\right)^2\theta(\frac{\Lambda^2}{P^+}-|k_1^- -k_3^- -q^-|)\theta(\frac{\Lambda^2}{P^+}-|k_4^- -k_2^- -q^-|)\\
&\times\left\{\frac{\theta(|k_1^- -k_3^- -q^-|-\frac{\lambda^2}{P^+})\theta(|k_1^- -k_3^- -q^-|-|k_4^- -k_2^- -q^-|)}{k_1^- -k_3^- -q^-}\right.\\
&\left.+\frac{\theta(|k_4^- -k_2^- -q^-|-\frac{\lambda^2}{P^+})\theta(|k_4^- -k_2^- -q^-|-|k_1^- -k_3^- -q^-|)}{k_4^- -k_2^- -q^-}\right\}\\
&=\theta(\frac{\lambda^2}{P^+}-|\Delta P_{ij}^-|)2(2\pi)^3\delta^3(\bar{k}_1+\bar{k}_2-\bar{k}_3-\bar{k}_4)\delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4}F(x-x',\kappa_\perp-\kappa'_\perp,M_H)\\
&\times\frac{4g^2(T^a)(T^a)}{(P^+)^2}\frac{1}{(x-x')^2}\frac{(\kappa_\perp-\kappa'_\perp)^2}{(\kappa_\perp-\kappa'_\perp)^2+(x-x')^2M_H^2},
\end{aligned} \tag{3.13}$$

where k^- is given by eq.(3.7), $q^+=k_1^+-k_3^+=k_4^+-k_2^+$, $q_\perp=k_{1\perp}-k_{3\perp}=k_{4\perp}-k_{2\perp}$, and $q^-=q_\perp^2/q^+$, we have also introduced the longitudinal residual momentum fractions and the relative transverse residual momenta,

$$\begin{aligned}
x&=k_1^+/P^+\ , \qquad \kappa_\perp=k_{1\perp}-xP_\perp,\\
x'&=k_3^+/P^+\ , \qquad \kappa'_\perp=k_{3\perp}-x'P_\perp,
\end{aligned} \tag{3.14}$$

and defined the function F,

$$F(x,k,M)\equiv\theta(A(x,k,M)-\lambda^2)\theta(\Lambda^2-A(x,k,M)), \tag{3.15}$$

with

$$A(x,k,M)\equiv\frac{k^2}{|x|}+|x|M^2. \tag{3.16}$$

Since $0\le p_1^+=m_Qv^++k_1^+\le P^+=M_Hv^+$, in the heavy quark mass limit, we have $M_H\to 2m_Q$ so that $-m_Qv^+\le k_1^+,\ k_3^+\le m_Qv^+$. Hence, the range of x and x' is given by

$$-\frac{1}{2}\le x,\ x'\le\frac{1}{2}. \tag{3.17}$$

Eqs.(3.11), (3.12) and (3.13) consist of the effective Hamiltonian for quarkonia up to the second order in the similarity renormalization group scheme.

Apparently, the above effective Hamiltonian is not a low energy Hamiltonian because $P^+ = M_H v^+$ which is of order a few GeV. However, from eq.(3.5) and eq.(3.8) we see that in the bare heavy quark Hamiltonian the mass term $m_Q v^-$ has been integrated out. To address the correct energy scale of the low energy heavy hadron dynamics, we should introduce the residual center mass momentum of the heavy quarkonia K^μ and the residual heavy hadron mass $\overline{\Lambda}$,

$$K^\mu = \overline{\Lambda} v^\mu \quad , \qquad \overline{\Lambda} = M_H - m_Q - m_{\overline{Q}}, \tag{3.18}$$

where v^μ is still the four-velocity of hadrons. It follows that

$$K^+ = k_1^+ + k_2^+ = k_3^+ + k_4^+ \quad , \quad K_\perp = k_{1\perp} + k_{2\perp} = k_{3\perp} + k_{4\perp}. \tag{3.19}$$

With the residual heavy hadron momentum K^μ considered, eqs.(3.12) and (3.13) become

$$\begin{aligned} H^{(2)}_{\lambda 1} = \theta(\frac{\lambda^2}{K^+} - |\Delta K^-_{ij}|)[2(2\pi)^3]^2 \delta^3(\bar{k}_1 - \bar{k}_3)\delta^3(\bar{k}_2 - \bar{k}_4)\delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4} \\ \times \frac{-8g^2 C_f}{K^+}\int \frac{dy_1 d^2\kappa_{1\perp}}{2(2\pi)^3}\frac{\theta(y-y_1)}{(y-y_1)^2}F(y-y_1, \kappa_\perp - \kappa_{1\perp}, \overline{\Lambda}) \\ \times \frac{(\kappa_\perp - \kappa_{1\perp})^2}{(\kappa_\perp - \kappa_{1\perp})^2 + (y-y_1)^2\overline{\Lambda}^2}, \end{aligned} \tag{3.20}$$

$$\begin{aligned} H^{(2)}_{\lambda 2} = \theta(\frac{\lambda^2}{K^+} - |\Delta K^-_{ij}|)2(2\pi)^3\delta^3(\bar{k}_1 + \bar{k}_2 - \bar{k}_3 - \bar{k}_4)\delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4}F(y-y', \kappa_\perp - \kappa'_\perp, \overline{\Lambda}^2) \\ \times \frac{4g^2(T^a)(T^a)}{(K^+)^2}\frac{1}{(y-y')^2}\frac{(\kappa_\perp - \kappa'_\perp)^2}{(\kappa_\perp - \kappa'_\perp)^2 + (y-y')^2\overline{\Lambda}^2}, \end{aligned} \tag{3.21}$$

where we have also introduced the corresponding residual relative momenta:

$$\begin{aligned} y = k_1^+/K^+ \ , \qquad \kappa_\perp = k_{1\perp} - yK_\perp, \\ y' = k_3^+/K^+ \ , \qquad \kappa'_\perp = k_{3\perp} - y'K_\perp. \end{aligned} \tag{3.22}$$

The range of the residual longitudinal momentum fractions y and y' are given by

$$-\infty < y = \frac{M_H}{\overline{\Lambda}}x < \infty \quad , \quad -\infty < y' = \frac{M_H}{\overline{\Lambda}}x' < \infty. \tag{3.23}$$

Now all the quantities appearing in the effective Hamiltonian have the low energy scale of a few MeV.

C. Reexpression of the low energy Hamiltonian in the weak-coupling treatment scheme

As we have mentioned in the Introduction, the first step to follow the idea of the weak-coupling approach is to construct the low energy Hamiltonian H_λ up to the second order and then separate it into $H_{\lambda 0}$ and $H_{\lambda I}$,

$$H_\lambda = H_{\lambda 0} + H_{\lambda I}. \tag{3.24}$$

In eq.(3.24), $H_{\lambda 0}$ contains the free Hamiltonian plus the dominant two-body interactions which conserve the particle number, and $H_{\lambda I}$ is the remaining interaction Hamiltonian which describes the emission and reabsorption processes plus all the higher order terms in the expansion of eq.(2.1). *Once H_λ is derived, we can reexpress it as eq.(3.24) such that $H_{\lambda 0}$ is set up to nonperturbatively determine the hadronic bound states, and $H_{\lambda I}$ should be treated perturbatively.* This separation is a basic step to realize a weak-coupling treatment of nonperturbative QCD [1]. Thus, besides the free quark and gluon Hamiltonian with constituent masses, $H_{\lambda 0}$ also contains the instantaneous interaction and all the second order contributions generated by integrating over all the modes above the low energy cutoff λ, namely, eqs.(3.20) and (3.21). Explicitly,

$$H_{\lambda 0ij} = \theta(\lambda^2/K^+ - |\Delta K^-_{ij}|)\Big\{H_{Q\overline{Q}free} + V_{Q\overline{Q}I}\Big\}, \tag{3.25}$$

where

$$\begin{aligned} H_{Q\overline{Q}free} =& [2(2\pi)^3]^2\delta^3(\bar{k}_1-\bar{k}_3)\delta^3(\bar{k}_2-\bar{k}_4)\delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4} \\ &\times\left\{k_1^- + k_2^- + \frac{\vec{k}_1^2}{2m_Q} + \frac{\vec{k}_2^2}{2m_{\overline{Q}}} - 2\frac{g^2}{4\pi^2}C_f\frac{\lambda^2}{K^+}\ln\epsilon\right\}, \end{aligned} \tag{3.26}$$

$$\begin{aligned} V_{Q\overline{Q}}(y-y',\kappa_\perp-\kappa'_\perp) =& 2(2\pi)^3\delta^3(\bar{k}_1+\bar{k}_2-\bar{k}_3-\bar{k}_4)\delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4}\frac{-4g^2(T^a)(T^a)}{(K^+)^2}\Bigg\{\frac{1}{(y-y')^2} \\ &+\frac{1}{(y-y')^2}\frac{\kappa_\perp-\kappa'_\perp)^2}{(\kappa_\perp-\kappa'_\perp)^2+(y-y')^2\overline{\Lambda}^2}\theta(A(y-y',\kappa_\perp-\kappa'_\perp,\overline{\Lambda})-\lambda^2)\Bigg\} \\ =& 2(2\pi)^3\delta^3(\bar{k}_1+\bar{k}_2-\bar{k}_3-\bar{k}_4)\delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4} \\ &\times\frac{-4g^2(T^a)(T^a)}{(K^+)^2}\Bigg\{\frac{1}{(y-y')^2}\Big(1-\theta(A(y-y',\kappa_\perp-\kappa'_\perp,\overline{\Lambda})-\lambda^2)\Big) \end{aligned}$$

$$+\frac{\overline{\Lambda}^2}{(\kappa_\perp-\kappa'_\perp)^2+(y-y')^2\overline{\Lambda}^2}\theta(A(y-y',\kappa_\perp-\kappa'_\perp,\overline{\Lambda})-\lambda^2)\Big\}. \qquad (3.27)$$

In (3.26), k_i^- is given by (3.7), $\vec{k}_i^2=\frac{2}{v^+}(k_\perp^2-\frac{1}{v^+}2v_\perp\cdot k_\perp k^+ +\frac{v^-}{v^+}k^{+2})$. In the above results, we have already let UV cutoff parameter $\Lambda\to\infty$ and the associated divergence has been put in the mass correction. The kinetic energy (3.10) now is included in the above nonperturbative part of the effective Hamiltonian.

The free energy part $H_{Q\overline{Q}free}$ of eq.(3.26) has also included the self-energy correction which is the instantaneous interaction contribution, a normal ordering term of the instantaneous interaction in (3.11), plus the one-loop contribution (3.20). The result is

$$\begin{aligned}\Sigma=4g^2C_f\int\frac{dy_1d^2\kappa_{1\perp}}{2(2\pi)^3}\Big\{&\frac{\theta(y-y_1)}{(y-y_1)^2}\\ &-\frac{\theta(y-y_1)}{(y-y_1)^2}\frac{(\kappa_\perp-\kappa_{1\perp})^2}{(\kappa_\perp-\kappa_{1\perp})^2+(y-y_1)^2\overline{\Lambda}^2}F(y-y_1,\kappa_\perp-\kappa_{1\perp},\overline{\Lambda})\Big\}\\ \overset{\Lambda\to\infty}{=}&\ 4g^2C_f\int\frac{dy_1d^2\kappa_{1\perp}}{2(2\pi)^3}\frac{\theta(y-y_1)}{(y-y_1)^2}\theta(\lambda^2-A(y-y_1,\kappa_\perp-\kappa_{1\perp},\overline{\Lambda}))+\delta m_Q^2\\ =&-\frac{g^2}{4\pi^2}\lambda^2C_f\ln\epsilon+\delta m_Q^2, \qquad (3.28)\end{aligned}$$

where the mass correction $\delta m_Q^2=\frac{g^2}{4\pi^2}C_f\overline{\Lambda}^2\ln\frac{\Lambda^2}{\lambda^2}$ which has been renormalized away in eq.(3.26). By removing away this mass correction, we should assign the corresponding constituent quark mass in $H_{\lambda 0}$ being λ-dependent. But, the heavy quark mass is much larger than the low energy scale. Its dependence on λ should be very weak and could be neglected.

The $Q\overline{Q}$ interaction $V_{Q\overline{Q}}$ of eq.(3.27) contains the one gluon exchange interaction eq.(3.21) plus the instantaneous interaction [the last term in (3.11)]. It clearly shows that without the low energy cutoff ($\lambda=0$), the instantaneous interaction is completely cancelled by the same contribution from the one transverse gluon exchange and the remaining one gluon exchange interaction is a Coulomb interaction, like in QED. Now, with the low energy cutoff, the one gluon exchange contribution only contains these gluons with the energy greater than the cutoff λ. Thus, the resulting $Q\overline{Q}$ interaction has two terms: The first term is the result of the noncancellation between the instantaneous interaction and one trans-

verse gluon exchange interaction, which corresponds to a confining potential. The second term, the rest of one transverse gluon exchange interaction, is the Coulomb interaction on the light-front. The detailed confinement mechanism on the light-front will be discussed in section V. With the kinetic energy incorporated, we see that the above effective QCD Hamiltonian which will be used to determine the heavy quarkonium bound states only has the spin symmetry but no flavour symmetry.

Before ending this section, we may compare the present formulation for heavy quarkonia with the nonrelativistic QCD formulation developed by Lepage et al. [9].

In the nonrelativistic QCD formulation, heavy quarkonia are described by an effective field theory of QCD in the nonrelativistic limit plus a systematic computations of the relativistic corrections (in terms of momentum scales $(Mv)^2$ and $(Mv^2)^2$) and QCD short distance corrections (in terms of the scale M^2). The nonperturbative QCD scale Λ^2_{QCD} is implicatively included in this formulation. Heavy quarkonium annihilation and production processes can then be factorized with respect of the above different scales. The dominant contributions in quarkonium processes, namely, the nonperturbative QCD dynamics of quarkonium bound states may be computed in lattice simulations.

Our formulation is based on the factorization scale m_Q which naturally separates the QCD short distance and long distance dynamics. The long distance dynamics is described by the residual momentum which is now controlled by $\Lambda^2_{QCD}/\lambda^2$ via the effective Hamiltonian $H_{\lambda 0}$. The resulting effective Hamiltonian derived from QCD by the similarity renormalization group approach contains explicitly the confining and Coulomb interactions which have encompassed the necessary long distance effects for heavy quarkonia. The quarkonium bound states can then be directly solved in the corresponding light-front bound state equation (as we shall see later). The short distance dynamics can be systematically computed in the ordinary perturbation theory. These include the QCD radiative corrections (controlled by $\ln(m_Q/\Lambda)$, where Λ is an UV cutoff), and the $1/m_Q$ corrections (controlled by $\overline{\Lambda}/m_Q$). Our formulation is fully relativistic. It is a more complete QCD formulation for heavy quarkonia in comparison to nonrelativistic QCD [9]. It allows to directly compute the nonperturbative

QCD dynamics without the help of lattice simulation. Moreover, it is also straightforward to extend this formulation to the heavy hadron system which contains a single heavy quark, as we shall see the next.

D. Low-energy effective Hamiltonian for heavy-light quark systems

The heavy-light quark system (heavy hadrons with one heavy quark) is one of the most interesting topics in the current study of heavy hadron physics. We now apply the similarity renormalization group approach to such system.

The bare cutoff Hamiltonian we begin with for heavy-light quark system is the combination of the heavy quark effective Hamiltonian (3.3) and the full Hamiltonian for the light quark (2.9). Due to the HQET, we may also introduce the residual center mass momentum for heavy-light systems,

$$K^+ = \overline{\Lambda} v^+ = p_1^+ + k_1^+ = p_2^+ + k_2^+, \qquad K_\perp = \overline{\Lambda} v_\perp = p_{1\perp} + k_{1\perp} = p_{2\perp} + k_{2\perp}, \tag{3.29}$$

where $\overline{\Lambda} = M_H - m_Q$, p_1 and p_2 are the light antiquark momenta in the initial and final states respectively, and k_1 and k_2 the residual momenta of the heavy quarks. The initial and final states in $Q\overline{q}$ sector are denoted by $|i\rangle = b_v^\dagger(k_1,\lambda_1)d^\dagger(p_1,\lambda_1')|0\rangle$ and $|j\rangle = b_v^\dagger(k_2,\lambda_2)d^\dagger(p_2,\lambda_2')|0\rangle$, respectively. The residual longitudinal momentum fractions and the residual relative transverse momenta are defined in the similar way as in quarkonium system,

$$\begin{aligned} y &= p_1^+/K^+ \ , \qquad \kappa_\perp = p_{1\perp} - yK_\perp, \\ y' &= p_2^+/K^+ \ , \qquad \kappa_\perp' = p_{2\perp} - y'K_\perp, \end{aligned} \tag{3.30}$$

but the range of the longitudinal momentum fractions y and y' is different:

$$0 < y = \frac{M_H}{\overline{\Lambda}}\frac{p_1^+}{P^+} < \infty \ \ , \ \ 0 < y' = \frac{M_H}{\overline{\Lambda}}\frac{p_2^+}{P^+} < \infty. \tag{3.31}$$

Following the general procedure described in the previous subsection, it is easy to find that the nonperturbative part of the low-energy effective Hamiltonian for heavy-light quark systems, which is given by

$$H_{\lambda 0ij} = \theta(\frac{\lambda^2}{K^+} - |\Delta K^-_{ij}|)\Big\{H_{Q\overline{q}free} + V_{Q\overline{q}I}\Big\}, \tag{3.32}$$

where

$$\begin{aligned}
H_{Q\overline{q}free} =& [2(2\pi)^3]^2\delta^3(\bar{k}_1 - \bar{k}_2)\delta^3(\bar{p}_1 - \bar{p}_2)\delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4} \\
&\times \left\{\frac{p_\perp^2 + m_q^2}{p^+} + \frac{1}{K^+}\Big(2K_\perp\cdot k_\perp - K^- k^+\Big) - \frac{g^2}{2\pi^2}C_f\frac{\lambda^2}{K^+}\ln\epsilon\right\}, \qquad (3.33)\\
V_{Q\overline{q}}(y-y', \kappa_\perp - \kappa'_\perp) =& \, 2(2\pi)^3\delta^3(\bar{k}_1 + \bar{p}_1 - \bar{k}_2 - \bar{p}_2)\delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4} \\
&\times (-2g^2)(T^a)(T^a)\left\{2\left(\frac{1}{q^+}\right)^2 + \frac{1}{q^+}\left(\frac{q_\perp^{i'}}{q^+} - \frac{v^{i'}}{v^+}\right)\left(2\frac{q_\perp^{i'}}{q^+} - \frac{p_{1\perp}^{i'}}{p_1^+} - \frac{p_{2\perp}^{i'}}{p_2^+}\right)\right. \\
&\times \theta(\frac{\Lambda^2}{K^+} - |p_1^- - p_2^- - q^-|)\theta(\frac{\Lambda^2}{K^+} - |k_2^- - k_1^- - q^-|) \\
&\times \left[\frac{\theta(|p_1^- - p_2^- - q^-| - \frac{\lambda^2}{K^+})\theta(|p_1^- - p_2^- - q^-| - |k_2^- - k_1^- - q^-|}{p_1^- - p_2^- - q^-}\right. \\
&\left.\left. + \frac{\theta(|k_2^- - k_1^- - q^-| - \frac{\lambda^2}{K^+})\theta(|k_2^- - k_1^- - q^-| - |p_1^- - p_2^- - q^-|}{k_2^- - k_1^- - q^-}\right]\right\} \\
\overset{\Lambda\to\infty}{=}& \, 2(2\pi)^3\delta^3(\bar{k}_1 + \bar{p}_1 - \bar{k}_2 - \bar{p}_2)\delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4}\frac{-2g^2(T^a)(T^a)}{(K^+)^2} \\
&\times \left\{\frac{2}{(y-y')^2} - \left[2\frac{(\kappa_\perp - \kappa'_\perp)^2}{(y-y')^2} - \frac{\kappa_\perp^2 - \kappa_\perp\cdot\kappa'_\perp}{y(y-y')} - \frac{\kappa_\perp\cdot\kappa'_\perp - (\kappa')_\perp^2}{y'(y-y')}\right]\right. \\
&\times \left[\frac{\theta(B-\lambda^2)\theta(B-A)}{(\kappa_\perp - \kappa'_\perp)^2 - \frac{y-y'}{y}(\kappa_\perp^2 + m_q^2) - \frac{y-y'}{y'}((\kappa')_\perp^2 + m_q^2)}\right. \\
&\left.\left. + \frac{\theta(A-\lambda^2)\theta(A-B)}{(\kappa_\perp - \kappa'_\perp)^2 + (y-y')^2\overline{\Lambda}^2}\right]\right\}, \qquad (3.34)
\end{aligned}$$

with

$$B \equiv \left|\frac{(\kappa_\perp - \kappa'_\perp)^2}{y-y'} - \frac{\kappa_\perp^2 + m_q^2}{y} - \frac{(\kappa')_\perp^2 + m_q^2}{y'}\right|, \tag{3.35}$$

$$A \equiv (\kappa_\perp - \kappa'_\perp)^2/|y-y'| + |y-y'|\overline{\Lambda}^2. \tag{3.36}$$

Here we do not include the heavy quark kinetic energy into $H_{\lambda 0}$ since the dominant kinetic energy is given by the constituent light quark. The heavy quark kinetic energy can be treated as a perturbative correction to $H_{\lambda 0}$. The heavy quark free energy has been written in (3.33) by

$$k^- = \frac{1}{v^+}\Big(2v_\perp\cdot k_\perp - v^- k^+\Big) = \frac{1}{K^+}\Big(2K_\perp\cdot k_\perp - K^- k^+\Big). \tag{3.37}$$

The low energy heavy-light quark effective Hamiltonian is m_Q-independent. It obviously has the spin and flavour symmetry, namely, the heavy quark symmetry. Comparing to the quarkonium systems, the $Q\overline{q}$ interactions are much more complicated. But it is not difficult to see that the above H_λ contains a confining potential.

Finally, it is also straightforward to extend the above derivation to light-light quark systems. The result is just eq.(2.19) but in terms of the relative momenta:

$$\begin{aligned} x = p_1^+/P^+ \quad , \quad & \kappa_\perp = p_{1\perp} - xP_\perp, \\ x' = p_3^+/P^+ \quad , \quad & \kappa'_\perp = p_{1\perp} - x'P_\perp. \end{aligned} \tag{3.38}$$

Here there is no residual center mass momentum for light-light system. The hadron momentum P^μ is already of order a low energy scale. We shall not intend to discuss the light-light systems in details in this paper. As we have pointed out in the Introduction, for the light-light quark systems, besides the confinement, chiral symmetry breaking also plays an essential role in the low energy hadronic dynamics. We shall remain the light-light quark systems for further investigation when we attempt to address the problem of chiral symmetry in light-front QCD [18].

In conclusion, we have obtained in this section the renormalized low energy effective QCD Hamiltonian for heavy-heavy and heavy-light quark systems, and extracted the non-perturbative part $H_{\lambda 0}$, eqs.(3.25) and (3.32), in the weak-coupling treatment scheme. We are now ready to solve heavy hadrons on the light-front.

IV. LIGHT-FRONT HEAVY HADRON BOUND STATE EQUATIONS

In this section, based on the low-energy heavy quark effective Hamiltonian derived in the previous section, we shall construct light-front bound state equations in the weak-coupling treatment scheme.

A. General structure of light-front bound state equations in the weak-coupling treatment scheme

In general, a hadronic bound state on the light-front can be expanded in the Fock space composed of states with definite number of particles [19,20]. Formally, it can be expressed as follows

$$|\Psi(P^+,P_\perp,\lambda_s)\rangle=\sum_{n,\lambda_i}\int\Big(\prod_i[d^3\bar{p}]\Big)2(2\pi)^3\delta^3(\bar{P}-\sum_i\bar{p}_i)|n,\bar{p},\lambda_i\rangle\Phi_n(x_i,\kappa_{\perp i},\lambda_i), \tag{4.1}$$

where $P^+,P_\perp$ are its total longitudinal and transverse momenta respectively and λ_s its total helicity, $|n,\bar{p},\lambda_i\rangle$ is a Fock state consisting of n constituents, each of which carries momentum $\bar{p}_i$ and helicity λ_i ($\sum_i\lambda_i=\lambda_s$); $\Phi(x_i,\kappa_{\perp i},\lambda_i)$ the corresponding amplitude which depends on the helicities λ_i, the longitudinal momentum fractions x_i, and the relative transverse momenta $\kappa_{\perp i}$:

$$x_i=\frac{p_i^+}{P^+}\quad,\quad \kappa_{i\perp}=p_{i\perp}-x_iP_\perp. \tag{4.2}$$

The eigenstate equation that the wave functions obey on the light-front is obtained from the operator Einstein equation $P^2=P^+P^--P_\perp^2=M^2$:

$$H_{LF}|P^+,P_\perp,\lambda_s\rangle=\frac{P_\perp^2+M^2}{P^+}|P^+,P_\perp,\lambda_s\rangle, \tag{4.3}$$

where $H_{LF}=P^-$ is the light-front Hamiltonian. Explicitly, for a meson wave function, the corresponding light-front bound state equation is:

$$\Big(M^2-\sum_i\frac{\kappa_{i\perp}^2+m_i^2}{x_i}\Big)\begin{bmatrix}\Phi_{q\bar{q}}\\ \Phi_{q\bar{q}g}\\ \vdots\end{bmatrix}=\begin{bmatrix}\langle q\bar{q}|H_{int}|q\bar{q}\rangle & \langle q\bar{q}|H_{int}|q\bar{q}g\rangle & \cdots\\ \langle q\bar{q}g|H_{int}|q\bar{q}\rangle & \cdots & \\ \vdots & \ddots & \end{bmatrix}\begin{bmatrix}\Phi_{q\bar{q}}\\ \Phi_{q\bar{q}g}\\ \vdots\end{bmatrix}, \tag{4.4}$$

where H_{int} is the interaction part of H_{LF}.

Obviously, solving eq.(4.4) from QCD with the entire Fock space is impossible. A basic motivation of introducing the weak-coupling treatment scheme is to simplify the complexities in solving the above equation. The Hamiltonian H_λ derived in the previous sections has

already decoupled the high and low energy states. Here we only consider the low energy states (hadronic bound states). Due to the kinematic feature of boost symmetry on the light-front, we can assign a relative small longitudinal light-front momentum to the bound states. On the other hand, the light-front infrared divergences force us to introduce a small cutoff on the longitudinal light-front momentum to each individual constituent. Thus, the hadronic bound states can only consist of the Fock space sectors with a few particles. This is a kinematic truncation on eqs.(4.1) and (4.4). Furthermore, the most important point in the weak-coupling treatment scheme is the reseparation of the Hamiltonian $H_\lambda = H_{\lambda 0} + H_{\lambda I}$. As we mentioned before, $H_{\lambda 0}$ which conserves particle number devotes to a nonperturbative evaluation to the bound states through eq.(4.4). And $H_{\lambda I}$ which describes the particle emissions and reabsoptions is hopefully a perturbative term in the weak-coupling treatment so that we may not consider its contribution to eq.(4.4). Then, eq.(4.4) becomes diagonal in Fock space with respect to the different particle number sectors,

$$\Big(M^2 - \sum_i \frac{\kappa_{i\perp}^2 + m_i^2}{x_i}\Big) \begin{bmatrix} \Phi_{q\overline{q}} \\ \Phi_{q\overline{q}g} \\ \vdots \end{bmatrix} = \begin{bmatrix} \langle q\overline{q}|H_{\lambda 0}|q\overline{q}\rangle & 0 & 0 \\ 0 & \langle q\overline{q}g|H_{\lambda 0}|q\overline{q}g\rangle & 0 \\ 0 & 0 & \ddots \end{bmatrix} \begin{bmatrix} \Phi_{q\overline{q}} \\ \Phi_{q\overline{q}g} \\ \vdots \end{bmatrix}. \qquad (4.5)$$

Now we see that the bound state equation is manable.

The second important step in the weak-coupling treatment to the low energy QCD is the use of a constituent picture. The success of the constituent quark model suggests that we may only consider the valence quark Fock space in determining the hadronic bound states from $H_{\lambda 0}$. In this picture, quarks and gluons must have constituent masses. This constituent picture can naturally be realized on the light-front [1]. However, an essential difference from the phenomenological constituent quark model description is that the constituent masses introduced here are λ dependent. This cutoff dependence of constituent masses (as well as the effective coupling constant) is determined by solving the bound states equation and fitting the physical quantities with experimental data. This is indeed a renormalization condition in nonperturbative QCD. Note that unlike the usual renormalization scheme in QED, quarks and gluons in QCD are not physically observable particles so that we can only

determine their renormalized masses and coupling constant in hadronic (composite particles) sectors. Once the constituent picture is introduced, we can truncate the general expression of the light-front bound states to only including the valence quark Fock space. The higher Fock space contributions can be calculated as a perturbative correction through $H_{\lambda I}$. Thus, for mesons, eq.(4.1) is reduced to the following simple form:

$$|\Psi(P^+,P_\perp,\lambda_s)\rangle = \sum_{\lambda_1\lambda_2}\int [d^3\bar{p}_1][d^3\bar{p}_2]2(2\pi)^3\delta^3(\bar{P}-\bar{p}_1-\bar{p}_2) \times\Phi_{q\overline{q}}(x,\kappa_\perp\lambda_1,\lambda_2)|q(p_1,\lambda_2)\overline{q}(p_2,\lambda_2)\rangle, \tag{4.6}$$

where $|q(p_1,\lambda_1)\overline{q}(p_2,\lambda_2)\rangle = b^\dagger(p_1,\lambda_1)d^\dagger(p_2,\lambda_2)|0\rangle$, and $b^\dagger, d^\dagger$ should be regarded as the creation operator of the constituent quark and antiquark respectively. Consequently, the constituent quark and gluon masses m_i and coupling constant g in the effective Hamiltonian $H_{\lambda 0}$ become explicit functions of the low energy cutoff λ.

It is worth pointing out that spin is always a troublesome issue in the light-front approach. The meson light-front bound state we have constructed is labelled by helicity rather than spin. However in practice low-energy hadronic states with definite spins are needed. This discrepancy is usually remedied by introducing the so-called Melosh rotation [21], which transforms a single particle state from the light-front helicity basis to the ordinary spin basis,

$$R(x_i,k_\perp,m_i) = \frac{m_i + x_iM_0 - i\sigma\cdot(\mathbf{n}\times\kappa_\perp)}{\sqrt{(m_i+x_iM_0)^2+\kappa_\perp^2}}, \tag{4.7}$$

where $\mathbf{n}=(0,0,1)$, and

$$M_0^2 = \frac{\kappa_\perp^2+m_1^2}{x} + \frac{\kappa_\perp^2+m_2^2}{1-x}. \tag{4.8}$$

With Melosh transformation incorporated, the light-front meson bound state with a definite spin can be expressed in the weak-coupling treatment scheme as follows

$$|\Psi(P^+,P_\perp,J,J_z)\rangle = \sum_{\lambda_1\lambda_2}\int [d^3\bar{p}_1][d^3\bar{p}_2]2(2\pi)^3\delta^3(\bar{P}-\bar{p}_1-\bar{p}_2) \times\Phi^{JJ_z}_{q\overline{q}}(x,\kappa_\perp,\lambda_1,\lambda_2)|q(p_1,\lambda_1)\overline{q}(p_2,\lambda_2)\rangle, \tag{4.9}$$

where

$$\Phi^{JJ_z}_{q\overline{q}}(x,\kappa_\perp,\lambda_1,\lambda_2)=\phi_{q\overline{q}}(x,\kappa_\perp)R^{JJ_z}_{\lambda_1\lambda_2}(x,\kappa_\perp), \tag{4.10}$$

$$R^{JJ_z}_{\lambda_1\lambda_2}(x,\kappa_\perp)=\sum_{s_1s_2}\langle\lambda_1|R^\dagger(x,\kappa_\perp,m_1)|s_1\rangle\langle\lambda_2|R^\dagger(1-x,-\kappa_\perp,m_2)|s_2\rangle\langle\frac{1}{2}s_1\frac{1}{2}s_2|JJ_z\rangle, \tag{4.11}$$

and $\langle\frac{1}{2}s_1\frac{1}{2}s_2|JJ_z\rangle$ is the Clebsch-Gordon coefficient. The normalization condition for the state $|\Psi(v,J,J_z)\rangle$ is taken to be

$$\langle\Psi(P'^+,P'_\perp,J',J'_z)|\Psi(P^+,P_\perp,J,J_z)\rangle=2(2\pi)^3P^+\delta^3(\bar{P}'-\bar{P})\delta_{J'J}\delta_{J'_zJ_z}, \tag{4.12}$$

which leads to

$$\int\frac{dxd^2\kappa_\perp}{2(2\pi)^3}|\phi_{q\overline{q}}(x,\kappa_\perp)|^2=1. \tag{4.13}$$

After the above consideration of the spin property on the light-front, eq.(4.4), for mesons, becomes a light-front Bethe-Salpeter equation:

$$\begin{aligned}\Big(M^2-M_0^2\Big)\Phi^{JJ_z}_{q\overline{q}}(x,\kappa_\perp,\lambda_1,\lambda_2)=&\left(-\frac{g_\lambda^2}{2\pi^2}\lambda^2C_f\ln\epsilon\right)\Phi^{JJ_z}_{q\overline{q}}(x,\kappa_\perp,\lambda_1,\lambda_2)\\&+\sum_{\lambda'_1\lambda'_2}\int\frac{dx'd^2\kappa'_\perp}{2(2\pi)^3}V_{eff}(x,\kappa_\perp,\lambda_1,\lambda_2;x',\kappa'_\perp,\lambda'_1,\lambda'_2)\\&\times\Phi^{JJ_z}_{q\overline{q}}(x',\kappa'_\perp,\lambda'_1,\lambda'_2),\end{aligned} \tag{4.14}$$

where V_{eff} is the effective $q\overline{q}$ interactions in eq.(2.19).

Melosh transformation is exact only for free theory. With interactions incorporated, the use of Melosh transformation is only an approximation. This approximation may be reasonably good for the lowest spin bound states, such as the lowest-lying scalar and vector mesons, since the wavefunction (the valence quark amplitude in the light-front bound states) for them is a scale function to the rotational transformation. In other words, the "orbit" angular momentum which is dynamically dependent in the light-front formulation may not contribute to the total spin of these lowest spin hadrons. Moreover, in this paper, we focus on heavy quark systems. As we shall see next, due to the heavy quark spin symmetry, Melosh transformation results in the exact spin structure with fixed parity for scalar and vector heavy mesons.

B. Bound state equation for heavy quarkonia

In this section, we shall explicitly consider the heavy quarkonium states. First of all, for quarkonia, the wave function (4.9) can be further simplified, especially for its spin structure due to the spin symmetry in HQET. Within the framework of light-front HQET, eq.(4.9) in the heavy quark limit is reduced to:

$$|\Psi(K,J,J_z)\rangle = \sum_{\lambda_1\lambda_2} R^{JJ_z}_{\lambda_1\bar{\lambda}_2} \int [d^3\bar{k}_1][d^3\bar{k}_2] 2(2\pi)^3\delta^3(\bar{K}-\bar{k}_1-\bar{k}_2) \times\phi_{Q\overline{Q}}(y,\kappa_\perp)|b^\dagger_v(k_1,\lambda_1)d^\dagger_v(k_2,\lambda_2)\rangle. \tag{4.15}$$

Here the wavefunction $\phi_{Q\overline{Q}}(y,\kappa_\perp)$ may be mass dependent due to the kinetic energy in $H_{\lambda 0}$ [see (3.26)]. The Melosh transformation matrix element (4.11) in quarkonium states becomes a pure kinematic factor,

$$R^{00}_{\lambda_1\lambda_2} = \frac{v^+}{2\sqrt{2}}\ \overline{u}(v,\lambda_1)\gamma^5 v(v,\lambda_2) \tag{4.16}$$

for a pseudoscalar meson, and

$$R^{1J_z}_{\lambda_1\lambda_2} = -\frac{v^+}{2\sqrt{2}}\ \overline{u}(v,\lambda_1)\,\not{\epsilon}(J_z) v(v,\lambda_2) \tag{4.17}$$

for vector mesons. The light-front spinors for heavy quarks are given by

$$u(v,\lambda) = \Big(1+\frac{\alpha\cdot v_\perp+\beta}{v^+}\Big)w_\lambda = \begin{pmatrix} 1 \\ \frac{1}{v^+}(\tilde{\sigma}\cdot v_\perp + i)\end{pmatrix}\chi_\lambda,$$
$$v(v,\lambda) = \Big(1+\frac{\alpha\cdot v_\perp-\beta}{v^+}\Big)w_{-\lambda} = \begin{pmatrix} 1 \\ \frac{1}{v^+}(\tilde{\sigma}\cdot v_\perp - i)\end{pmatrix}\chi_{-\lambda}, \tag{4.18}$$

so that

$$\overline{u}(v,\lambda)u(v,\lambda') = \frac{2}{v^+}\delta_{\lambda\lambda'}\quad,\quad \sum_\lambda u(v,\lambda)\overline{u}(v,\lambda) = \frac{1+\ \not{v}}{v^+}, \tag{4.19}$$

$$\overline{v}(v,\lambda)v(v,\lambda') = -\frac{2}{v^+}\delta_{\lambda\lambda'}\quad,\quad \sum_\lambda v(v,\lambda)\overline{v}(v,\lambda) = -\frac{1-\ \not{v}}{v^+}, \tag{4.20}$$

and the polarization vector is defined by

$$\epsilon^\mu(\pm 1) = \Big(\frac{2}{v^+}\epsilon_\perp\cdot v_\perp, 0, \epsilon_\perp\Big)\ ,\quad \epsilon^\mu(0) = -\Big(\frac{v_\perp^2-1}{v^+}, v^+, v_\perp\Big)\ ,\quad \epsilon_\perp(\pm 1) = \mp\frac{1}{\sqrt{2}}(1\pm i). \quad (4.21)$$

Thus we have constructed the light-front heavy quarkonium bound states in the heavy mass limit, which have definite spin and parity. The corresponding spin tensor structures are given by eqs.(4.16) and (4.17).

Note that the heavy quarkonium states in heavy mass limit are labelled by the residual center mass momentum K^μ. We may normalize eq.(4.15) as follows:

$$\langle\Psi(K',J',J'_z)|\Psi(K,J,J_z)\rangle = 2(2\pi)^3K^+\delta^3(\bar{K}-\bar{K}')\delta_{J'J}\delta_{J'_zJ_z}, \quad (4.22)$$

which leads to

$$\int\frac{dyd^2\kappa_\perp}{2(2\pi)^3}|\phi_{q\overline{q}}(y,\kappa_\perp)|^2 = 1. \quad (4.23)$$

With the quarkonium states given above, it is easy to derive the corresponding bound state equation. In the weak-coupling scheme, $H_{LF} = H_{\lambda 0}$ where $H_{\lambda 0}$ is given by (3.25). Thus, the quarkonium bound state equation in light-front HQET is given by

$$(K^- - H_{\lambda 0})|\Psi(P,J,J_z)\rangle = 0. \quad (4.24)$$

The free energy part of the quarkonia states is extremely simple,

$$K^- - k_1^- - k_2^- = \frac{1}{K^+}2\overline{\Lambda}^2 \equiv \frac{1}{K^+}(\overline{\Lambda}^2 - \overline{M}_0^2), \quad (4.25)$$

where $\overline{M}_0^2 = -\overline{\Lambda}^2$ is a residual invariant mass (=the invariant mass M_0^2 subtracted by the mass dependent terms, here $\overline{M}^2$ is just a notation rather than a real square of a quantity). It follows that eq.(4.24) can be expressed explicitly by

$$\begin{aligned}\Big\{2\overline{\Lambda}^2 - \frac{\overline{\Lambda}}{m_Q}\big[2\kappa_\perp^2 + \overline{\Lambda}^2(2y^2-2y+1)\big]\Big\}\phi_{Q\overline{Q}}(y,k_\perp) &= \Big(-\frac{g_\lambda^2}{2\pi^2}\lambda^2 C_f\ln\epsilon\Big)\ \phi_{Q\overline{Q}}(y,k_\perp)\\ &- 4g_\lambda^2(T^a)(T^a)\int\frac{dy'd^2\kappa'_\perp}{2(2\pi)^3}\Big\{\frac{1}{(y-y')^2}\theta(\lambda^2-A)\\ &+\frac{\overline{\Lambda}^2}{(\kappa_\perp-\kappa'_\perp)^2+(y-y')^2\overline{\Lambda}^2}\theta(A-\lambda^2)\Big\}\phi_{Q\overline{Q}}(y',\kappa'_\perp). \quad (4.26)\end{aligned}$$

This is the bound state equation for heavy quarkonia in the weak-coupling treatment of low energy QCD.

C. Bound state equation for heavy-light quark systems

In the previous subsection, we have derived explicitly the bound state equation for quarkonia. In the last few years, the heavy-light quark systems have been extensively explored theoretically and experimentally. The discovery of the heavy quark symmetry in the heavy mass limit [10] allows one to extract the Kobayashi-Maskawa matrix element $|V_{cb}|$ without knowing the detailed structure of the heavy-light mesons. However, to have a complete description for various heavy hadron processes, one has to know a number of heavy hadron matrix elements involved the heavy hadron bound states. Currently, most of heavy hadron matrix elements have only been calculated in various phenomenological models, such as quark models, QCD sum rules etc. It is necessary to find these heavy-light hadron bound states from the fundamental QCD. In this subsection, we shall derive from the low energy effective QCD Hamiltonian the bound state equation obeyed by the heavy mesons with one heavy quark.

For the heavy mesons with one heavy quark, due to the heavy quark symmetry the general form of the wavefunction in the weak-coupling scheme can also be simplified. Considering the heavy quark mass limit, the wavefunction (4.9) can be expressed as

$$|\Psi(K,J,J_z)\rangle = \sum_{\lambda_1\lambda_2}\int [d^3\bar{k}][d^3\bar{p}]2(2\pi)^3\delta^3(\bar{K}-\bar{k}-\bar{p}) \times R^{JJ_z}_{\lambda_1\lambda_2}(y,\kappa_\perp)\phi_{Q\overline{Q}}(y,\kappa_\perp)|b^\dagger_v(k,\lambda_1),d^\dagger(p,\lambda_2)\rangle. \tag{4.27}$$

Here the Melosh transformation matrix element, eq.(4.11), is a little more complicated in comparing to the heavy quarkonium state:

$$R^{00}_{\lambda_1\lambda_2} = \frac{1}{2}\sqrt{\frac{p^+K^+}{2(y\overline{\Lambda}^2+\frac{\kappa_\perp^2+m_q^2(\lambda)}{y})}}\,\overline{u}(v,\lambda_1)\gamma^5 v(p,\lambda_2) \tag{4.28}$$

for a pseudoscalar meson, and

$$R^{1J_z}_{\lambda_1\lambda_2} = -\frac{1}{2}\sqrt{\frac{p^+K^+}{2(y\overline{\Lambda}^2+\frac{\kappa_\perp^2+m_q^2(\lambda)}{y})}}\,\overline{u}(v,\lambda_1)\not{\epsilon}(J_z)v(p,\lambda_2) \tag{4.29}$$

for vector mesons. The light-front spinors for the light antiquark is

$$v(p,\lambda)=\Big(1+\frac{\alpha\cdot p_\perp-\beta m_q(\lambda)}{p^+}\Big)w_{-\lambda}=\begin{pmatrix}1\\ \frac{1}{p^+}(\tilde{\sigma}\cdot p_\perp-im_q(\lambda))\end{pmatrix}\chi_{-\lambda}, \tag{4.30}$$

and the polarization vector ϵ^μ is still given by eq.(4.21).

Eq.(4.27) is a light-front heavy-light meson bound state in the symmetry limit ($m_Q\to\infty$), which has the definite spin and parity. The corresponding bound state equation then becomes

$$\begin{aligned}\Big(\overline{\Lambda}^2+(1-y)\overline{\Lambda}^2-\frac{\kappa_\perp^2+m_q^2(\lambda)}{y}\Big)&\Phi^{JJ_z}_{Q\overline{q}}(y,k_\perp,\lambda_1,\lambda_2)\\ &=\Big(-\frac{g_\lambda^2}{2\pi^2}\lambda^2C_f\ln\epsilon\Big)\ \Phi^{JJ_z}_{Q\overline{q}}(y,k_\perp,\lambda_1,\lambda_2)\\ &\quad+(K^+)^2\int\frac{dy'd^2\kappa'_\perp}{2(2\pi)^3}V_{Q\overline{q}}(y-y',\kappa_\perp-\kappa'_\perp)\Phi^{JJ_z}_{Q\overline{q}}(y',\kappa'_\perp,\lambda_1,\lambda_2),\end{aligned} \tag{4.31}$$

where $V_{Q\overline{q}}$ is given by eq.(3.34),

$$\Phi^{JJ_z}_{Q\overline{q}}(y,\kappa_\perp,\lambda_1,\lambda_2)=\phi_{Q\overline{q}}(y,\kappa_\perp)R^{JJ_z}_{\lambda_1\lambda_2}(y,\kappa_\perp), \tag{4.32}$$

and $R^{JJ_z}_{\lambda_1\lambda_2}$ is determined by eq.(4.28) or (4.29). Here the light antiquark is a brown muck, a current quark surround by infinite gluons and $q\overline{q}$ pairs that result a constituent quark mass which is a function of λ.

Thus, we have derived in this section the bound state equations in the weak-coupling scheme of the non-perturbative QCD for the light-light, heavy-heavy and heavy-light mesons. By solving these equations and comparing with experimental data, such as meson mass spectroscopy, we can determine the λ dependence of the constituent quark masses, the effective coupling constant as well as the wavefunction renormalization (anomalous dimensions) of hadronic states. Then we are able to use the corresponding wavefunctions to describe and predict various hadronic processes. The low energy cutoffs, or more precisely, the low energy scale dependences indeed reveal the inherent QCD dynamics of hadronic bound states. For completeness, we should also derive the bound state equation for glueball states (gg bound states). The glueball bound state equation is not only the basis for the study of the currently searching glueball states, but also allow us to determine another very important quantity in

the present weak-coupling treatment of low energy QCD, i.e., the constituent gluon mass and its scale dependence. But in this paper, we shall mainly consider heavy hadron systems. As we have seen, in the heavy quarkonium bound states, the constituent light quark and gluon masses do not appear. We only need to determine the effective coupling constant. Thus, the quarkonium states are the simplest systems in present theory. In fact, the determination of the scale dependence of the effective coupling constant g_λ is the most important problem, from which we can test whether a weak-coupling treatment of nonperturbative QCD can be realized in this framework. In the following sections, we shall study heavy quarkonia in details. The extension to heavy-light quark systems will be briefly discussed and the more explicit study will be presented in the forthcoming publication.

Before proceeding to solve the bound state equations derived in this section, we shall demonstrate first from these bound state equations how quark confinement is realized on the light-front.

V. QUARK CONFINEMENT ON THE LIGHT-FRONT

In the conventional picture, QCD has a complex vacuum that contains infinite quark pairs and gluons necessary for confinement and chiral symmetry breaking. On the light-front, the longitudinal momentum of physical particles is always positive, $p^+ = p^0 + p^3 \geq 0$. As a result, only these constituents with zero longitudinal momentum (called zero modes) can occupy the light-front vacuum. The zero modes carry an extremely high light-front energy which has been integrated out in the similarity renormalization group scheme. Consequently, some equivalent effective interactions associated with the effect of the nontrivial QCD vacuum are generated in the low energy Hamiltonian H_λ. Furthermore, the use of the constituent picture in the weak-coupling scheme forbids possible occurrence of any zero modes in H_λ in the subsequent computations. Therefore, light-front QCD vacuum remains trivial. The nature of nontrivial QCD vacuum structure, the confinement as well as the chiral symmetry breaking must be made manifestly in H_λ in terms of new effective interactions. In fact,

upon to the second order calculation in the similarity renormalization scheme (see sections II and III), the effective Hamiltonian H_λ already contains a confining interaction. The interactions associated with the chiral symmetry breaking may be manifested in the fourth order computation of H_λ, as pointed out by Wilson [15], but these interactions are not important in the study of heavy hadrons. Hence, next we shall only discuss the quark confinement in terms of the light-front bound state equations, from which a light-front picture of confinement mechanism becomes transparent.

To be specific, we take the following criteria as a definition of quark confinement: i) No color non-singlet bound states exist in nature, only color singlet states with finite masses can be produced and observed; ii) There is a confining potential for quark interaction such that quarks cannot be well-separated; iii) The conclusions of i–ii) are only true for QCD but not for QED. If conditions i–iii) could be verified from the low energy effective QCD Hamiltonian $H_{\lambda 0}$ and the corresponding bound state equations, then quark confinement is realized on the light-front. Here we shall take heavy quarkonia as an explicit example. Some of the ideas for this confinement picture have been discussed in [4,15].

In the present formulation of low-energy QCD, non-existence of color non-singlet bound states is essentially related to infrared divergences in the effective Hamiltonian. First of all, we shall show how only for physical states the infrared divergence in the quark self-energy correction is cancelled exactly by the same divergence from the uncancelled instantaneous interaction in eq.(4.26). It is obvious that when $y' \to y$, the uncancelled instantaneous interaction leads to a severe infrared divergence. Assuming that $\phi_{Q\overline{Q}}(y, \kappa_\perp)$ is a smooth function with respect to y and $\kappa_\perp$, and vanishes when $y' \to \infty$. Then the dominant contribution of the following integral is given by,

$$
\begin{aligned}
&- 4g_\lambda^2 (T^a)(T^a) \int \frac{dy' d^2\kappa'_\perp}{2(2\pi)^3} \frac{1}{(y-y')^2} \theta(\lambda^2 - A(y-y', \kappa_\perp - \kappa'_\perp, \overline{\Lambda})) \phi_{Q\overline{Q}}(y, \kappa'_\perp) \\
&\quad \sim 4g_\lambda^2 (T^a)(T^a) \phi_{Q\overline{Q}}(y, \kappa_\perp) \int_{y-\epsilon}^{y+\epsilon} \frac{dy' d^2\kappa'_\perp}{2(2\pi)^3} \frac{1}{(y-y')^2} \theta(\lambda^2 - A(y-y', \kappa_\perp - \kappa'_\perp, \overline{\Lambda})) \\
&\quad = \left(\frac{g_\lambda^2 \lambda^2}{2\pi^2} (T^a)(T^a) \ln \epsilon \right) \phi_{Q\overline{Q}}(y, k_\perp). \qquad (5.1)
\end{aligned}
$$

A more complete computation with an explicit light-front wavefunction will be given in the next section.

Eq.(5.1) indicates that in bound state equations, the uncancelled instantaneous interaction contains a logarithmic infrared divergence. Except for the color factor, this infrared divergence has the same form as the divergence in the self-energy correlation. From the bound state equation (4.26), we immediately obtain the following conclusions.

(a). For a single (constituent) quark state, the bound state equation simply leads to

$$\overline{\Lambda}^2 = -\frac{g_\lambda^2 \lambda^2}{4\pi^2} C_f \ln \epsilon. \tag{5.2}$$

This means that mass correction for single quark states is infinite (infrared divergent) and cannot be renormalized away in the spirit of gauge invariance. Equivalently speaking, single quark states carry an infinite mass and therefore they cannot be produced.

(b). For color non-singlet composite states, the color factor $(T^a)_{\alpha\beta}(T^a)_{\delta\gamma}$ in the $Q\overline{Q}$ interaction is different from the color factor $\delta_{\alpha\beta} C_f = (T^a T^a)_{\alpha\beta}$. Therefore, the infrared divergence in the self-energy correction also cannot be cancelled by the corresponding divergence from the uncancelled instantaneous interaction. As a result, color non-singlet composite states are infinitely heavy so that they cannot be produced as well.

(c). For color singlet $Q\overline{Q}$ states, the color factor $(T^a)(T^a)$ becomes $(T^a T^a) = C_f$. Thus, the infrared divergences in eq.(4.26) are completely cancelled and the binding energy of the corresponding bound states is guaranteed to be finite. In other words, only color singlet composite particles are physically observable bound states, as a solution of eq.(4.26).

The above conclusion is also true for heavy-light and light-light hadronic states [see eqs.(4.14) and (4.31)]. This provides the first condition for quark confinement in light-front QCD. Indeed, the physical origin of the above result is very clear. Light-front infrared divergences are associated with violation of gauge invariance. Only in gauge noninvariant sectors, light-front infrared divergences may occur. In gauge invariant sectors, infrared divergences must be automatically cancelled. Therefore, the above conclusion is a natural consequence of gauge invariance.

Physically, in order to be consistent with the above conclusion, confinement must also imply the existence of a confining potential so that quarks cannot be well-separated to become asymptotically free states. Now we can show that *the interactions in effective Hamiltonian (3.25) contains indeed both a confining and a Coulomb potentials.*

The Coulomb potential can be easily obtained by applying the Fourier transformation to the second term in (3.27). It is convenient to perform the calculation in the frame $K_\perp = 0$, in which

$$\begin{aligned}&\overline{\Lambda}^2 \frac{1}{(\kappa_\perp - \kappa'_\perp)^2 + (y-y')^2\overline{\Lambda}^2} \\ &\quad \sim \frac{1}{4\pi}\int dx^- d^2x_\perp e^{i(x^- q^+ + q_\perp \cdot x_\perp)} \left(\frac{\overline{\Lambda}}{K^+}\right) \frac{1}{\sqrt{x_\perp^2 + \left(\frac{\overline{\Lambda}}{K^+}\right)^2 (x^-)^2}},\end{aligned} \tag{5.3}$$

where $q^+ = k_1^+ - k_3^+ = K^+(y-y'), q_\perp = k_{1\perp} - k_{3\perp} = \kappa_\perp - \kappa'_\perp$ for $K_\perp = 0$. Eq.(5.3) shows that the Coulomb potential on the light-front for quarkonium states has the form

$$V_{Coul.}(x^-, x_\perp) = -\frac{g_\lambda^2}{4\pi} C_f \frac{\overline{\Lambda}}{K^+} \frac{1}{\sqrt{x_\perp^2 + \left(\frac{\overline{\Lambda}}{K^+}\right)^2 (x^-)^2}} = -\frac{g_\lambda^2}{4\pi} C_f \frac{\overline{\Lambda}}{K^+} \frac{1}{r_l}, \tag{5.4}$$

where

$$r_l = \sqrt{x_\perp^2 + \left(\frac{\overline{\Lambda}}{K^+}\right)^2 (x^-)^2} \tag{5.5}$$

is defined to be a "radial" variable in light-front space [1].

The confining potential corresponds to the finite part of the non-cancelled instantaneous interaction in (3.27). Its Fourier transformation is relatively complicated. The general expression is

$$\begin{aligned}&\int \frac{dq^+ d^2q_\perp}{(2\pi)^2} e^{i(q^+x^- + q_\perp \cdot x_\perp)} \left\{ -\frac{4g_\lambda^2 C_f}{K^{+2}} \frac{1}{(y-y')^2} \theta(\lambda^2 - A(y-y', \kappa_\perp - \kappa'_\perp, \overline{\Lambda})) \right\} \\ &\qquad = -\frac{g_\lambda^2}{2\pi^2} C_f \int_0^{\frac{\lambda^2}{\overline{\Lambda}^2} K^+} dq^+ e^{iq^+x^-} \frac{q_{\perp m}^2}{q^{+2}} \frac{2J_1(|x_\perp| q_{\perp m})}{|x_\perp| q_{\perp m}},\end{aligned} \tag{5.6}$$

where $q_{\perp m} = \sqrt{\frac{\lambda^2}{K^+} q^+ - \frac{\overline{\Lambda}^2}{K^{+2}} q^{+2}}$, and $J_1(x)$ is a Bessel function. An analytic solution to the integral (5.6) may be difficult to carry out. However, the nature of confining interactions is

a large distance QCD dynamics. We may consider the integral for large x^- and $x_\perp$. In this case, if q^+x^- and/or $|x_\perp|q_{\perp m}$ are large, the integration vanishes, yet $J_1(x) = \frac{x}{2} + \frac{x^3}{16} + \cdots$ for small x. The dominant contribution of the integral (5.6) for large x^- and $x_\perp$ comes from the small q^+ such that q^+x^- and/or $|x_\perp|q_{\perp m}$ must remain small and therefore

$$e^{iq^+x^-}\frac{2J_1(|x_\perp|q_{\perp m})}{|x_\perp|q_{\perp m}} \sim 1. \tag{5.7}$$

This corresponds to

$$q^+ < \frac{1}{x^-} \quad \text{and/or} \quad q^+ < \frac{K^+}{|x_\perp|^2\lambda^2}. \tag{5.8}$$

If $q^+ < \frac{1}{x^-} < \frac{K^+}{|x_\perp|^2\lambda^2}$, eq.(5.6) is reduced to

$$-\frac{g_\lambda^2}{2\pi^2}C_f\int_0^{\frac{1}{x^-}} dq^+\frac{1}{q^{+2}}\left(\frac{\lambda^2}{K^+}q^+ - \frac{\overline{\Lambda}^2}{K^{+2}}q^{+2}\right) = \frac{g_\lambda^2\lambda^2}{2\pi^2K^+}C_f\Big(\ln|x^-| + \ln\epsilon\Big), \tag{5.9}$$

where a term $\sim \frac{1}{x^-}$ is neglected since x^- is large, and ϵ is an infrared cutoff on q^+. The infrared logarithmic divergence ($\sim \ln\epsilon$) exactly cancels the divergence in the self-energy corrections in H_λ, so that the remaining is a logarithmic confining potential:

$$V_{conf.}(x^-, x_\perp) = \frac{g_\lambda^2\lambda^2}{2\pi^2K^+}C_f\ln|x^-|. \tag{5.10}$$

Similarly, when $q^+ < \frac{K^+}{|x_\perp|^2\lambda^2} < \frac{1}{x^-}$, we have

$$-\frac{g_\lambda^2}{2\pi^2}C_f\int_0^{\frac{K^+}{|x_\perp|^2\lambda^2}} dq^+\frac{1}{q^{+2}}\Big(\frac{\lambda^2}{K^+}q^+ - \frac{\overline{\Lambda}^2}{K^{+2}}q^{+2}\Big) = \frac{g_\lambda^2\lambda^2}{2\pi^2K^+}C_f\Big(\ln\frac{\lambda^2|x_\perp|^2}{K^+} + \ln\epsilon\Big), \tag{5.11}$$

where the term $\sim \frac{1}{x_\perp^2}$ has also been ignored because of large $x_\perp^2$. Again, the infrared divergence ($\sim \ln\epsilon$) is cancelled in H_λ, and we obtain the following confining potential:

$$V_{conf.}(x^-, x_\perp) = \frac{g_\lambda^2\lambda^2}{2\pi^2K^+}C_f\ln\frac{\lambda^2|x_\perp|^2}{K^+}. \tag{5.12}$$

Thus, the effective Hamiltonian $H_{\lambda 0}$ contains a logarithmic confining potential in all the directions of x^- and $x_\perp$ coordinates. Note that for heavy quarkonia, a logarithmic confining potential provides indeed a good description to the spectroscopy and leptonic decays [22]. More details of computation will be given in the next section. Nevertheless, we have explicitly shown here that $H_{\lambda 0}$ exhibits a Coulomb potential at short distances and a confining

potential at long distances. The second condition for quark confinement is verified on the light-front.

Finally, we shall argue that the above mechanism of quark confinement is indeed only true for QCD. As we have seen the light-front confinement potential is just an effect of the non-cancellation between instantaneous interaction and one transverse gluon interaction. Such a non-cancellation arises from the requirement in the similarity renormalization group scheme that the transverse gluon energy cannot be below a certain value (about a few MeV). This requirement is naturally satisfied if the gluon mass is nonzero in low energy scale. Unlike the constituent quark mass which we know is an effect of the spontaneous chiral symmetry breaking, the origin of constituent massive gluons is not very clear at present. The assumption of massive gluons here may also violate gauge invariance but it is not unnatural. In fact, if gluon were massless like photons, the hadronic spectra would be continuous rather than discrete, as Wilson pointed out recently [15]. A typical evidence of gluons being massive in the low energy domain is the possible existence of glueball states which is still a very active topic in current experimental searches [23]. The massive gluon must be originated from the nonlinear interactions in non-abelian gauge theory. Therefore, the non-cancellation of the instantaneous interaction in the low energy domain is indeed a consequence of the existence of the constituent massive gluons due to the non-abelian gauge interactions. This is independent of any particular renormalization scheme. The use of the low energy cutoff λ just gives us a simple realization of this confining picture that the massive gluon exchange energy cannot run down to the zero value in nonperturbative QCD. In the case of lacking the mechanism of how the massive gluons are generated, the determination of the gluon mass lies on the solution of bound state equations.

Based on the above discussion, it is now easy to find that the confinement mechanism described in this section is not valid in QED. First of all, the infrared divergence in the self energy is also a result of the noncancellation between the instantaneous interaction self-energy diagram and the one-loop self-energy corrections [see eq.(3.28)]. In QED, since the photon mass is always zero, the photon energy in the one-loop self-energy correction

covers the entire range from zero to infinity. Thus, in QED, we can always choose the low energy cutoff λ being zero. (We shall further explain in Section VII that this is indeed the only choice for applying similarity renormalization group approach to QED. Otherwise the resulting effective QED theory is inconsistent with the perturbative QED theory.) Then a direct calculation for (3.28) with $\lambda = 0$ shows that the infrared divergences do not occur in the electron self-energy correction. As a result, the renormalized single electron mass is finite, in contrast to the divergent mass of single quark states. For the same reason, with $\lambda = 0$, the instantaneous interaction in the effective QED Hamiltonian is also exactly cancelled by the same interaction from one transverse photon exchange so that only one photon exchange Coulomb interaction remains. Therefore, using similarity renormalization group approach to QED, we obtain a conventional effective QED Hamiltonian which only contains the Coulomb interaction. Such an effective Hamiltonian is the basis in the study of positronium bound states. More discussion will be given in Section VII.

Now we shall study how a weak-coupling treatment scheme works in solving hadronic bound state problem in the present low energy QCD formulation.

VI. QCD DESCRIPTION OF QUARKONIA ON THE LIGHT-FRONT

A numerical computation to the heavy hadron bound state equations, eqs.(4.26) and (4.31), is actually not too difficult. However, to have a deeper insight about the internal structure of light-front bound states and to determine the scale dependence of the effective coupling constant in H_λ, it is better to have an analytic analysis. In this paper, the light-front wavefunction ansatz will be used to solve the bound state equations for heavy quarkonia, from which some general properties of the low energy scaling in the similarity renormalization group can be extracted. It also provides a direct test whether the weak-coupling treatment of nonperturbative QCD can be realized.

A. A general analysis of light-front wavefunctions

For heavy quark systems, the wavefunctions considered in the previous section are defined in heavy mass limit. Most of the $1/m_Q$ corrections can be handled in the standard perturbation theory in the present framework, except for the kinetic energy for quarkonia. The heavy hadronic wavefunctions in the heavy mass limit can be tremendously simplified.

First of all, the heavy quark kinematics have already added some constraints on the general form of the light-front wavefunction $\phi(x, \kappa_\perp)$. The kinetic energy part (the left hand side of these bound state equations in section IV) shows that when we introduce the residual longitudinal momentum fraction y for heavy quarks, the longitudinal momentum fraction dependence in ϕ is quite different for the heavy-heavy, heavy-light and light-light mesons.

For the light-light mesons, such as pions, rhos, kaons etc., the wavefunction $\phi_{q\overline{q}}(x, \kappa_\perp)$ must vanish at the endpoint $x = 0$ or 1. This can be seen from the kinetic energy contribution in the bound state equation [see eq.(4.14)],

$$M^2 - M_0^2 = M^2 - \frac{\kappa_\perp^2 + m_1^2}{x} - \frac{\kappa_\perp^2 + m_2^2}{1-x}. \tag{6.1}$$

To ensure that the bound state equation is well defined in the entire range of momentum space, $|\phi_{q\overline{q}}(x, \kappa_\perp)|^2$ must fall down to zero in the longitudinal direction not slower than $1/x$ and $1/(1-x)$ when $x \to 0$ and 1, respectively. In other words, at least $\phi_{q\overline{q}}(x, \kappa_\perp) \sim \sqrt{x(1-x)}$.

For heavy-light quark mesons, namely the B and D mesons, the wavefunction $\phi_{Q\overline{q}}(y, \kappa_\perp)$ is required to vanish at $y = 0$, where y is the residual longitudinal momentum fraction carried by the light quark. This is because the kinetic energy in eq.(4.31) contains a singularity at $y = 0$,

$$M^2 - M_0^2 \longrightarrow \overline{\Lambda}^2 - \overline{M}_0^2 = \overline{\Lambda}^2 + (1-y)\overline{\Lambda}^2 - \frac{\kappa_\perp^2 + m_1^2}{y}. \tag{6.2}$$

On the other hand, since $0 \le y \le \infty$, $\phi_{Q\overline{q}}(y, \kappa_\perp)$ should also vanish when $y \to \infty$. Hence, a possible simple form is $\phi_{Q\overline{q}}(y, \kappa_\perp) \sim \sqrt{y} e^{-\alpha y}$ or $\sqrt{y} e^{-\alpha y^2}$. The y dependence in $\phi_{Q\overline{q}}(y, \kappa_\perp)$ is obviously different from the x dependence in $\phi_{q\overline{q}}(x, \kappa_\perp)$.

For heavy quarkonia, the kinetic energy in the corresponding bound state equation (4.26) is:

$$M^2 - M_0^2 \longrightarrow \overline{\Lambda}^2 - \overline{M}_0^2 = 2\overline{\Lambda}^2 - \frac{\overline{\Lambda}}{m_Q}\Big[2\kappa_\perp^2 + \overline{\Lambda}^2(2y^2 - 2y + 1)\Big]. \tag{6.3}$$

Since $-\infty < y < \infty$, the normalization forces $\phi_{Q\overline{Q}}(y, \kappa_\perp)$ to vanish as $y \to \pm\infty$. Therefore a possible form is $\phi_{Q\overline{Q}}(y, \kappa_\perp) \sim e^{-\alpha y^2}$. Obviously, in heavy quark mass limit, the y dependence in $\phi_{Q\overline{Q}}(y, \kappa_\perp)$ is very different from the above two cases.

On the other hand, the transverse momentum dependence in these light-front wavefunctions should be more or less similar. They all vanish at $\kappa_\perp \to \pm\infty$. A simple form of the $\kappa_\perp$ dependence for these wavefunctions is a Gaussian function: $e^{-\kappa_\perp^2/2\omega^2}$.

The above analysis of light-front wavefunctions is only based on the kinetic energy properties of the constituents. Currently, many investigations on the hadronic structures use phenomenological light-front wavefunctions. One of such phenomenological wavefunctions that has been widely used in the study of heavy hadron structure is the so-called BSW wavefunction, introduced by Bauer et al. [24],

$$\phi_{BSW}(x, \kappa_\perp) = \mathcal{N}\sqrt{x(1-x)}\ \exp\left(-\frac{\kappa_\perp^2}{2\omega^2}\right)\ \exp\left[-\frac{M_H^2}{2\omega^2}(x - x_0)^2\right], \tag{6.4}$$

where $\mathcal{N}$ is a normalization constant, ω a parameter of order Λ_{QCD}, $x_0 = (\frac{1}{2} - \frac{m_1^2 - m_2^2}{2M_H^2})$, and M_H, m_1, and m_2 are the hadron, quark, and antiquark masses respectively. In the phenomenological description, the parameters ω, and m_i $(i = 1, 2)$ in (6.4) are fitted from data. Here we are of course interested in the dynamical determination of these parameters.

As we have pointed out in passing, for heavy quark systems, the $1/m_Q$ corrections can be well treated perturbatively in our framework (except for the kinetic energy of quarkonia). Here we are only interested in the solution of the wavefunctions in the heavy mass limit, where eq.(6.4) can be further simplified.

Explicitly, for heavy-light quark systems, such as the B and D mesons, one can easily find that in the heavy mass limit,

$$m_1 = m_Q \sim M_H, \quad m_q << m_Q \quad \text{so that} \quad x_0 = 0. \tag{6.5}$$

Meanwhile, from eq.(3.31), we also have

$$M_H x = \overline{\Lambda} y. \tag{6.6}$$

Furthermore, the factor $\sqrt{x(1-x)}$ can be rewritten by $\sqrt{y}$ in according to the discussion on eq.(6.2). Thus, the BSW wavefunction is reduced to

$$\phi_{Q\overline{q}}(y, \kappa_\perp) = \mathcal{N}\sqrt{y}\ \exp\left(-\frac{\kappa_\perp^2}{2\omega^2}\right)\ \exp\left(-\frac{\overline{\Lambda}^2}{2\omega^2}y^2\right). \tag{6.7}$$

This agrees with our qualitative analysis given before. Using such a wavefunction we have already computed the universal Isgur-Wise function in $B \to D, D^*$ decays [6],

$$\xi(v \cdot v') = \frac{1}{v \cdot v'}, \tag{6.8}$$

and from which we obtained the slope of $\xi(v \cdot v')$ at the zero-recoil point, $\rho^2 = -\xi'(1) = 1$, in excellent agreement with the recent CLCO result [26] of $\rho^2 = 1.01 \pm 0.15 \pm 0.09$. This result is independent of the value of ω, and therefore is independent of further dynamics of QCD involved in the corresponding bound state equation. The simple form (6.7) is just a consequence of heavy quark symmetry in the our low energy theory. We may argue that $\rho^2 = 1$ could be a universal identity.

For heavy quarkonia, such as the $b\overline{b}$ and $c\overline{c}$ states, $m_1 = m_2 = m_Q$ which leads to $x_0 = 1/2$ in eq.(6.4). Also, the longitudinal momentum fraction in (6.4) is defined by $x = p_1^+/P^+$, its relation to the residual longitudinal momentum fraction is given by

$$M_H(x - \frac{1}{2}) = \overline{\Lambda} y. \tag{6.9}$$

In addition, the factor $\sqrt{x(1-x)}$ must be totally dropped as we have seen from the discussion on eq.(6.3). Therefore the BSW wavefunction for quarkonia is reduced to

$$\phi_{Q\overline{Q}}(y, \kappa_\perp) = \mathcal{N}\ \exp\left(-\frac{\kappa_\perp^2}{2\omega^2}\right)\ \exp\left(-\frac{\overline{\Lambda}^2}{2\omega^2}y^2\right), \tag{6.10}$$

which is the exact form as we expected from the qualitative analysis. Here we have not taken the limit of $m_Q \to \infty$ for heavy quarkonia. Thus a possible m_Q dependence in wavefunction may be hidden in the parameter ω.

Another phenomenological light-front wavefunction which has been widely used for both light and heavy mesons has the form [25]

$$\psi_{q\overline{q}}(x,\kappa_\perp) = \mathcal{N}\sqrt{\frac{d\kappa_z}{dx}}\ \exp\left(-\frac{\kappa_\perp^2}{2\omega^2}\right)\ \exp\left(-\frac{\kappa_z^2}{2\omega^2}\right), \tag{6.11}$$

where κ_z is defined by

$$x = \frac{e_1+\kappa_z}{e_1+e_2}, \quad 1-x = \frac{e_2-\kappa_z}{e_1+e_2}; \quad e_i = \sqrt{\kappa_\perp^2+\kappa_z^2+m_i^2}\ \ (i=1,2) \tag{6.12}$$

as a pretended z-component of relative momentum while $\sqrt{d\kappa_z/dz}$ is the Jacobian of transformation from $(x,\kappa_\perp)$ to $\vec{\kappa} = (\kappa_\perp,\kappa_z)$. This wavefunction has been used frequently in various studies of hadronic transitions. In particular, it has been shown that this wavefunction describes satisfactorily the pion elastic form factor up to $Q^2 \sim 10$ GeV^2 [25].

For heavy quarkonia with $m_1 = m_2 = m_Q$, we may have

$$\kappa_z = M_0(x-\frac{1}{2}) \longrightarrow \overline{\Lambda} y. \tag{6.13}$$

Thus, $\sqrt{d\kappa_z/dx}$ = constant, and eq.(6.11) is reduced to the same form obtained from the BSW wavefunction,

$$\phi_{q\overline{q}}(x,\kappa_\perp) \longrightarrow \phi_{Q\overline{Q}}(y,\kappa_\perp) = \mathcal{N}\ \exp\left(-\frac{\kappa_\perp^2}{2\omega^2}\right)\ \exp\left(-\frac{\overline{\Lambda}^2}{2\omega^2}y^2\right). \tag{6.14}$$

Therefore, the above Gaussian-type ansatz should be a very good candidate for the low-lying quarkonium states. In the following, we start with this wavefunction ansatz to solve the light-front quarkonium bound state equation, and from which to determine the low energy scaling dynamics and develop the weak-coupling treatment of the heavy hadron bound states.

B. A weak-coupling realization of the nonperturbative QCD description for heavy quarkonia

Based on the analysis in the last section, we take the normalized wavefunction ansatz of (6.10),

$$\phi_{Q\overline{Q}}(y,\kappa_\perp)=4\sqrt{\overline{\Lambda}}\left(\frac{\pi}{\omega_\lambda^2}\right)^{3/4}\exp\left(-\frac{\kappa_\perp^2}{2\omega_\lambda^2}\right)\exp\left(-\frac{\overline{\Lambda}^2}{2\omega_\lambda^2}y^2\right), \tag{6.15}$$

as a solution (a trial wavefunction) of the heavy quarkonium bound state equation (4.26). Note that here we have also specified the scale dependence of the wavefunction through the scale dependence of the parameter ω_λ. Substituting the above wavefunction into the quarkonium bound state equation (4.26) and introducing the new variables

$$\begin{aligned} Z&=\frac{1}{2}(y+y'),\quad z=y-y',\\ Q_\perp&=\frac{1}{2}(\kappa_\perp+\kappa'_\perp),\quad q_\perp=\kappa_\perp-\kappa'_\perp, \end{aligned} \tag{6.16}$$

we have

$$\begin{aligned} 2\overline{\Lambda}^2=&\frac{\overline{\Lambda}}{m_Q}\left(3\omega_\lambda^2+\overline{\Lambda}^2\right)-\frac{g_\lambda^2}{2\pi^2}\lambda^2C_f\ln\epsilon\\ &-4g_\lambda^2C_f\int\frac{dzd^2q_\perp}{2(2\pi)^3}\exp\left\{-\frac{1}{4\omega_\lambda^2}(q_\perp^2+z^2\overline{\Lambda}^2)\right\}\\ &\times\left\{\frac{1}{z^2}\theta(\lambda^2|z|-q_\perp^2-z^2\overline{\Lambda}^2)+\frac{\overline{\Lambda}^2}{q_\perp^2+z^2\overline{\Lambda}^2}\theta(q_\perp^2+z^2\overline{\Lambda}^2-\lambda^2|z|)\right\}\\ =&\mathcal{E}_{kin}-\frac{g_\lambda^2}{2\pi^2}\lambda^2C_f\ln\epsilon+\mathcal{E}_{nonc}+\mathcal{E}_{Coul}, \end{aligned} \tag{6.17}$$

where $\mathcal{E}_{kin}$ represents the kinetic energy, $\mathcal{E}_{nonc}$ is the contribution of the noncancellation of the instantaneous interaction, and $\mathcal{E}_{Coul}$ from the Coulomb interaction. Furthermore, it is not very difficult to compute that

$$\begin{aligned} \mathcal{E}_{nonc}=&-4g_\lambda^2C_f\int\frac{dzd^2q_\perp}{2(2\pi)^3}\frac{1}{z^2}\theta(\lambda^2|z|-q_\perp^2-z^2\overline{\Lambda}^2)\exp\left\{-\frac{1}{4\omega_\lambda^2}(q_\perp^2+z^2\overline{\Lambda}^2)\right\}\\ =&\frac{g_\lambda^2}{2\pi^2}C_f\lambda^2\left\{\gamma+\ln\frac{\lambda^2\epsilon}{4\omega_\lambda^2}+\mathrm{E}_1(\varpi^2)+\frac{\sqrt{\pi}}{\varpi}\mathrm{Erf}(\varpi)\right\}, \end{aligned} \tag{6.18}$$

$$\begin{aligned} \mathcal{E}_{Coul}=&-4g_\lambda^2C_f\int\frac{dzd^2q_\perp}{2(2\pi)^3}\frac{\overline{\Lambda}^2}{q_\perp^2+z^2\overline{\Lambda}^2}\theta(q_\perp^2+z^2\overline{\Lambda}^2-\lambda^2|z|)\exp\left\{-\frac{1}{4\omega_\lambda^2}(q_\perp^2+z^2\overline{\Lambda}^2)\right\}\\ =&-\frac{g_\lambda^2}{2\pi^2}C_f\lambda^2\left\{\frac{\sqrt{\pi}}{\varpi}\left[1-\ \mathrm{Erf}(\varpi)\right]+\frac{1}{\varpi^2}\left[1-e^{-\varpi^2}\right]\right\}, \end{aligned} \tag{6.19}$$

where $\gamma=0.57721566...$ is the Euler constant, ϵ is the small longitudinal momentum cutoff, the dimensionless ϖ is defined by

$$\varpi = \frac{\lambda^2}{2\omega_\lambda \overline{\Lambda}}, \tag{6.20}$$

and E_1 and Erf are the exponential integral function and the error function, respectively,

$$\mathrm{E}_1(x) = \int_x^\infty \frac{e^{-t}}{t} dt \quad , \qquad \mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt. \tag{6.21}$$

We may rewrite the term $\ln \frac{\lambda^2 \epsilon}{4\omega_\lambda^2}$ in $\mathcal{E}_{nonc}$ as

$$\ln \frac{\lambda^2 \epsilon}{4\omega_\lambda^2} = \ln \epsilon + \ln \varpi^2 + \ln \frac{\overline{\Lambda}^2}{\lambda^2}. \tag{6.22}$$

It shows that $\mathcal{E}_{nonc}$ contains a logarithmic divergence $\ln \epsilon$ which exactly cancels the same divergence from the self-energy correction, as expected, and the term $\ln \varpi^2$ is the logarithmic confining energy.

After the cancellation of the infrared $\ln \epsilon$ divergences in eq.(6.17), the binding energy for heavy quarkonia is given by the kinetic energy plus the potential energy:

$$\begin{aligned} 2\overline{\Lambda}^2 &= \mathcal{E}_{kin} + \mathcal{E}_{conf} + \mathcal{E}_{Coul} \\ &= \frac{\overline{\Lambda}}{m_Q}\left\{3\omega_\lambda^2 + \overline{\Lambda}^2\right\} + \frac{g_\lambda^2}{2\pi^2} C_f \lambda^2 \left\{\gamma + \ln \varpi^2 + \ln \frac{\overline{\Lambda}^2}{\lambda^2} + E_1(\varpi^2) + \frac{\sqrt{\pi}}{\varpi}\mathrm{Erf}(\varpi)\right\} \\ &\qquad - \frac{g_\lambda^2}{2\pi^2} C_f \lambda^2 \left\{\frac{\sqrt{\pi}}{\varpi}\Big[1 - \mathrm{Erf}(\varpi)\Big] + \frac{1}{\varpi^2}\Big[1 - e^{-\varpi^2}\Big]\right\} \\ &= \frac{\overline{\Lambda}}{m_Q}\left\{3\omega_\lambda^2 + \overline{\Lambda}^2\right\} + \frac{g_\lambda^2}{2\pi^2} C_f \lambda^2 \left\{F(\varpi) + \ln \frac{\overline{\Lambda}^2}{\lambda^2}\right\}, \end{aligned} \tag{6.23}$$

where

$$F(\varpi) = \gamma + \ln \varpi^2 + E_1(\varpi^2) - \frac{\sqrt{\pi}}{\varpi}\Big[1 - 2\mathrm{Erf}(\varpi)\Big] - \frac{1}{\varpi^2}\Big[1 - e^{-\varpi^2}\Big], \tag{6.24}$$

which is a dimensionless function. In Fig.1, we plot the confining potential energy, the Coulomb potential energy and the totally potential energy as functions of ϖ which is proportional to the radial variable in light-front space,

$$\varpi \sim \frac{1}{\omega_\lambda} \sim r_l. \tag{6.25}$$

Fig.1 shows that *the total potential energy is a typical combination of the Coulomb potential in short distance and a confining potential in long distance that has been widely used in*

previous phenomenological describing hadronic states, but it is now explicitly derived from QCD. Furthermore, eq.(6.23) also indicates that without considering the kinetic energy, we cannot find stable quarkonium bound states. The kinetic energy balances the potential energy and ensures the existence of a stable solution for (6.23). Therefore, the first order kinetic energy in HQET is an important nonperturbative effect in binding two heavy quarks, as noticed first by Mannel et al. [17] in their attempt of applying HQET to heavy quarkonium system.

If we know the experimental value of the quarkonium binding energy $\overline{\Lambda}$, minimizing eq.(6.23) can completely determine the parameter ω_λ and the coupling constant g_λ. The precise value of quarkonium binding energy that can be compared with the data in Particle Data Group [27] must include the spin-splitting energy ($1/m_Q$ corrections) which we will present in the forthcoming paper [28]. Here, to justify whether a weak-coupling treatment of the nonperturbative QCD can become possible in the present formulation, we will give a schematic calculation. It is known that $\overline{\Lambda}$ is of the same order as Λ_{QCD} which is about $100 \sim 400$ MeV. To solve (6.23) we shall take several values of $\overline{\Lambda}$ within the above range. We choose the low energy cutoff about a typical hadronic energy, $\lambda = 1$ GeV. The charmed and bottom quark masses used here are $m_c = 1.4$ GeV and $m_b = 4.8$ GeV. The results are listed in Tables I and II for charmonium and bottomonium, respectively, where $\omega_{\lambda 0}$ denotes the minimum point of the binding energy (6.23).

We see from the Tables I–II that the coupling constant $\alpha_\lambda = g_\lambda^2/4\pi$ is very small. For instance, with $\overline{\Lambda} = 200$ MeV, we obtain

$$\alpha_\lambda = \begin{cases} 0.02665 & \text{charmonium}, \\ 0.06795 & \text{bottomonium}, \end{cases} \tag{6.26}$$

which is much smaller than that extrapolated from the running coupling constant in the naive perturbative QCD calculation. The parameter $\omega_{\lambda 0}$ in the quarkonium wavefunction is the mean value of the (transverse) momentum square of heavy quark inside the heavy quarkonia:

$$\langle k_\perp^2 \rangle = \omega_\lambda^2. \tag{6.27}$$

For charmonium, we can see that the resulting ω_λ are typical values of $\Lambda_{QCD} \sim \overline{\Lambda}$. The kinetic energy is about a half of the potential energy. For bottomonium, we find that the binding energy $\overline{\Lambda}$ cannot be too large. In fact, when $\overline{\Lambda}$ is over about 260 MeV, eq.(6.23) has no solution. Meanwhile, compared to charmonium, the effective coupling constant is relatively large (in contrast to the perturbative running coupling constant which is smaller with increasing m_Q if it is taken as the mass scale). Also the values of ω_λ in bottomonium wavefunctions are larger than that in charmonium. The difference between charmonium and bottomonium in the nonperturbative calculation may be understood as follows. As we know, in the nonrelativistic quark model, the quark momentum in quarkonia is proportional to the quark mass, $\omega_\lambda \sim m_Q$ [22]. Our relativistic QCD bound state solution exhibits such a property. This is why the values of ω_λ for bottomonium are much larger than that for charmonium. As a result, the bottomonium kinetic energy ($\sim \omega_\lambda^2$) becomes large as well. To have a nonperturbative balance between the kinetic energy and the potential energy in the bound states, the coupling constant in bottomonium must be larger than that in charmonium. All these properties now have been manifested in the solution of eq.(6.23). A more precise determination of α_λ (i.e., g_λ) requires an accurate computation of the low-lying quarkonium spectroscopy with the $1/m_Q$ corrections included [28]. Nevertheless, it has been shown that the effective coupling constant in the low energy Hamiltonian H_λ is very small at the hadronic mass scale.

In order to see how this weak coupling constant varies with the cutoff λ, we take $\overline{\Lambda} = 200$ MeV and vary the value of λ around 1 GeV. The result is listed in Table III. We find that the coupling constant is decreased very faster with increasing λ. In other words, *with a suitable choice of the low energy cutoff λ in the similarity renormalization group scheme, we can make the effective coupling constant α_λ in H_λ arbitrarily small, and therefore the weak-coupling treatment of the non-perturbative QCD can be achieved in terms of H_λ such that the corrections from $H_{\lambda I}$ can be computed perturbatively.* Thus, we have provided the

first explicit realization of recently proposed the weak-coupling treatment of nonperturbative QCD on the light-front [1].

We must emphasize here that α_λ is not the physical coupling constant $\alpha_s = g_s^2/4\pi$. The later is of order unity at the hadronic mass scale. A detailed analysis of the λ-dependence and the relation between α_λ and α_s will be discussed in the next.

VII. SIMILARITY RENORMALIZATION GROUP EQUATION AND LOW ENERGY RUNNING COUPLING CONSTANT

In this section, we shall discuss the scale dependence of the coupling constant, the constituent quark and gluon masses as well as the wavefunctions. For heavy quarkonia, the bound state equation does not include the constituent gluon mass. The heavy quark mass is larger than the usual hadronic mass scale. Its λ-dependence should be very weak that can be neglected in the present discussion. Thus the remainings are the λ-dependence of the coupling constant g_λ and the wavefunctions, the later is described through the λ-dependence of the parameter ω_λ.

From these solutions in Tables I to III, we find that the values of dimensionless parameter $\varpi = \frac{\lambda^2}{2\overline{\Lambda}\omega_{\lambda 0}}$ are greater than 2.5. When $x > 2.5$, the exponential integral function and the error function are simply reduced to $\mathrm{E}_1(x) = 0$ and $\mathrm{Erf}(x) = 1$. Thus, the dimensionless function $F(\varpi)$ can be expressed approximately by

$$F(\varpi) = \gamma + \ln \varpi^2 + \frac{\sqrt{\pi}}{\varpi} - \frac{1}{\varpi^2} \quad , \qquad \varpi \geq 2.5, \tag{7.1}$$

with an error less than 10^{-5}. Hence we can simply rewrite eq.(6.23) as

$$2\overline{\Lambda}^2 = \frac{3\overline{\Lambda}}{m_Q}\omega_\lambda^2 - \frac{g_\lambda^2}{2\pi^2}C_f\lambda^2\left\{\frac{4\overline{\Lambda}^2\omega_\lambda^2}{\lambda^4} - \sqrt{\pi}\,\frac{2\overline{\Lambda}\omega_\lambda}{\lambda^2} - \left(\gamma + \ln\frac{\lambda^2}{4} - \ln\omega_\lambda^2\right)\right\} + \frac{\overline{\Lambda}^3}{m_Q}. \tag{7.2}$$

Minimizing $\overline{\Lambda}$ with respect to ω_λ, we obtain

$$\left\{\frac{3\overline{\Lambda}}{m_Q} - \frac{g_\lambda^2}{2\pi^2}C_f\frac{4\overline{\Lambda}^2}{\lambda^2}\right\}\omega_\lambda^2 = \frac{g_\lambda^2}{2\pi^2}C_f\lambda^2\left\{1 - \sqrt{\pi}\,\frac{\overline{\Lambda}\omega_\lambda}{\lambda^2}\right\}. \tag{7.3}$$

Therefore, eq.(7.2) becomes

$$2\overline{\Lambda}^2 = \frac{g_\lambda^2}{2\pi^2} C_f \lambda^2 \left\{ 1 + \gamma + \ln\frac{\lambda^2}{4} - \ln\omega_{\lambda 0}^2 + \sqrt{\pi}\ \frac{\overline{\Lambda}\omega_{\lambda 0}}{\lambda^2} \right\} + \frac{\overline{\Lambda}^3}{m_Q}, \tag{7.4}$$

where $\omega_{\lambda 0}$ is a solution of (7.3). Eqs.(7.3) and (7.4) determine the λ-dependences of the coupling constant g_λ and the wavefunction parameter $\omega_{\lambda 0}$.

Directly and analytically solving eqs.(7.3) and (7.4) is not obviously possible. The nonperturbative balance between the kinetic energy and the potential energy implies that $\omega_\lambda \sim \sqrt{m_Q \overline{\Lambda}}$. Meanwhile, since it is the binding energy of heavy quarkonia, $\overline{\Lambda}$ should be λ-independent. We then obtain

$$\begin{aligned} \alpha_\lambda = \frac{g_\lambda^2}{4\pi} &= \frac{\pi}{C_f} \left(\frac{\overline{\Lambda}^2}{\lambda^2}\right) \left(1 - \frac{\overline{\Lambda}}{2m_Q}\right) \frac{1}{1 + \gamma + \ln\frac{\lambda^2}{4} - \ln\omega_{\lambda 0}^2 + \sqrt{\pi}\ \frac{\overline{\Lambda}\omega_{\lambda 0}}{\lambda^2}} \\ &= \frac{\pi}{C_f} \left(\frac{\overline{\Lambda}^2}{\lambda^2}\right) \frac{1}{a + b\ln\frac{\lambda^2}{\overline{\Lambda}^2}}, \end{aligned} \tag{7.5}$$

where the coefficients a and b can be obtained by numerically solving eqs.(7.3) and (7.4). The coefficient b is almost a constant (with a slight dependence on m_Q but independence on $\overline{\Lambda}$ and λ), while a depends on $\overline{\Lambda}, \lambda$ and also m_Q. For $\lambda \geq 0.6$ GeV, the λ-dependence of the parameter a is negligible. In Fig.2, we plot the λ-dependence of the effective coupling constant α_λ for charmonium. The dots are the numerical solutions of (6.23) and the solid line is given by the analytical form (7.5) with $b = 1.15$, and $a = -0.25$ for $\overline{\Lambda} = 0.2$ GeV and $a = 1.1$ for $\overline{\Lambda} = 0.4$ GeV. We can see that (7.5) is a very good analytical solution of the eqs.(7.3) and (7.4) [or of the minimizing eq.(6.23)].

The above solution shows that with increasing λ, α_λ becomes weaker and weaker. Meanwhile, we also find that the confining energy becomes more and more dominant in the binding energy (See Table IV). Fig.3 is a plot of the parameter $\omega_{\lambda 0}$ as a function of λ, from which we also see that with increasing λ, $\omega_{\lambda 0}$ is decreased. Correspondingly, the distance between two quarks inside the quarkonia, $r_l \sim \frac{1}{\omega_{\lambda 0}}$, is increased. This is why the confining interaction becomes more and more important. On the other hand, the confining interaction comes from the noncancellation of instantaneous gluon exchange with energy below the scale λ. With the larger λ, the more the instantaneous interaction contributes to H_λ. Thus, the above

conclusion can also be directly understood from the low energy Hamiltonian H_λ. Compared to the canonical QCD theory, the confining interaction should become more important if the scale Q^2 would be smaller, and correspondingly the running coupling constant $\overline{\alpha}(Q^2)$ becomes larger. This indicates that there is an inverse correspondence between the effective coupling constant α_λ in the low energy Hamiltonian H_λ and the running coupling constant $\overline{\alpha}(Q^2)$ in the full QCD theory:

$$\alpha_\lambda \sim \frac{1}{\overline{\alpha}(Q^2)} \ , \quad \text{and} \quad \lambda^2 \sim \frac{1}{Q^2}. \tag{7.6}$$

In other words, *the weak-coupling treatment of the low energy confining Hamiltonian H_λ may correspond to an inverse strong-coupling expansion of the full QCD theory. The similarity renormalization group approach provides an implicative realization for such an expansion.* This may be the inherent property why the nonperturbative QCD can be treated as weak-coupling problem in the similarity renormalization group scheme and why we can find the confining interaction in this weak-coupling QCD formulation.

We also find from Table IV that the confining interaction plays a more important role than the Coulomb interaction in the determination of the quarkonium bound states. This result is different from the usual understanding in the nonrelativistic phenomenological description that the dominant contribution in heavy quarkonium spectroscopy is the Coulomb interaction. This discrepancy can be understood as follows. The currently relativistic light-front description for heavy quark system mostly uses the heavy quark masses of $m_c = 1.3 \sim 1.4$ GeV and $m_b = 4.8$ GeV or less (In Particle Data Group [27], $m_b = 1.0 \sim 1.6$ GeV and $m_c = 4.1 \sim 4.5$ GeV). Thus, the heavy quarkonium binding energies, $\overline{\Lambda} = M_H - 2m_Q$, might be positive [the lowest charmonium ground state $M(\eta_c(1S)) = 2.98$ GeV, and the bottomonium $M(\Upsilon(1S)) = 9.46$ GeV]. Therefore, the Coulomb energy is obviously not important. The dominant contribution for binding quarkonium states must come from the nonperturbative balance between the kinetic energy and the confining energy. While, in the nonrelativistic phenomenological description, one used the larger quark masses, $m_c > 1.8$ GeV and $m_b > 5.1$ GeV [29], such that the binding energy is negative and therefore the

Coulomb interactions must be dominant in this picture. Of course, on the light-front, the structure of the bound state equation is different from the nonrelativistic Schrödinger equation. There is no direct comparison. A real solution to the above discrepancy may be obtained after including the spin-splitting interactions ($1/m_Q$-corrections).

Now we can study the running behavior of the coupling constant in the similarity renormalization group scheme. Denote

$$\overline{\Lambda} = \overline{\Lambda}(g_\lambda, \omega_\lambda, \lambda). \tag{7.7}$$

The invariance of the binding energy $\overline{\Lambda}$ under the similarity renormalization group transformation means that the $\overline{\Lambda}$ determined from H_λ and $H'_{\lambda'}$ must be the same for $\lambda \neq \lambda'$. Let $\lambda' = \lambda + \delta\lambda$, we obtain the corresponding similarity renormalization group equation

$$\Big(\lambda\frac{\partial}{\partial\lambda} + \beta\frac{\partial}{\partial g_\lambda} + \gamma_\omega\frac{\partial}{\partial\omega_\lambda}\Big)\overline{\Lambda}(g_\lambda, \omega_\lambda, \lambda) = 0, \tag{7.8}$$

where the quantity β is the similarity renormalization group β function which is defined by

$$\beta(g_\lambda) = \lambda\frac{dg_\lambda}{d\lambda}\bigg|_{\lambda=\lambda(g_\lambda)}, \tag{7.9}$$

and γ_ω is an anomalous dimension that describes the running properties of the bound state wavefunction. The β function can be computed from eq.(7.5),

$$\begin{aligned}\beta &= -g_\lambda\bigg(1 + \frac{2b}{a + 2b\ln\frac{\lambda}{\overline{\Lambda}}}\bigg)\bigg|_{\lambda=\lambda(g_\lambda)} \\ &\approx -g_\lambda \qquad \text{(for a relatively large} \quad \lambda >> \overline{\Lambda}).\end{aligned} \tag{7.10}$$

On the other hand, the running coupling constant in full QCD theory is given by

$$t = \int_{g_s}^{\overline{g}}\frac{dg'}{\beta(g')}, \tag{7.11}$$

where $t = \frac{1}{2}\ln\frac{Q^2}{\mu^2}$, and Q^2 is a space-like momentum (the same as λ^2). Since the similarity renormalization group β function of eq.(7.10) is determined in the physical sector of low energy QCD dynamics, the low energy β function of the running coupling constant $\overline{g}(Q^2)$ in the full theory should behave qualitatively the same. With this assumption, the $\beta(g)$

function in the above equation may take the same form as eq.(7.10) in low momentum transfer (namely $Q^2 << \Lambda^2_{QCD}$). This leads to

$$\overline{g}^2(Q^2) = g_s^2 \frac{\mu^2}{Q^2} \quad , \qquad Q^2 << \Lambda^2_{QCD}, \tag{7.12}$$

and $g_s^2 = g_s^2(\mu^2)$ is a fixed coupling constant at the hadronic mass scale μ^2. In terms of the running coupling constant $\alpha = g^2/4\pi$, we have

$$\overline{\alpha}(Q^2) = \alpha_s(\mu^2)\frac{\mu^2}{Q^2} \equiv c_0 \frac{\Lambda^2_{QCD}}{Q^2}, \tag{7.13}$$

where $c_0 = \alpha_s(\mu^2)\mu^2/\Lambda^2_{QCD}$. This is consistent with eq.(7.6). Furthermore, we see that the fixed point of the coupling constant under renormalization group transformation is the origin of g, and it is an infrared unstable (UV stable) fixed point, in consistence with the asymptotic freedom of QCD.

To give a qualitative determination of the coefficient c_0, we consider $\lambda = 0.75 \sim 1.5$ GeV and $\overline{\Lambda} = 0.2$ GeV, then $\lambda^2 = (14 \sim 56)\overline{\Lambda}^2 >> \overline{\Lambda}^2$. Rewriting (7.5) as the same form of (7.13), we obtain:

$$\alpha_\lambda = (1.0 \sim 1.5)\ \frac{\overline{\Lambda}^2}{\lambda^2}. \tag{7.14}$$

The corresponding $Q^2 \sim \frac{1}{\lambda^2} << \overline{\Lambda}^2 \sim \Lambda^2_{QCD}$. From (7.6), We may require that

$$\frac{\alpha_\lambda}{\overline{\alpha}(Q^2)} = \frac{Q^2}{\lambda^2}. \tag{7.15}$$

It follows that $c_0 = 1.0 \sim 1.5$, namely for $Q^2 << \Lambda^2_{QCD}$

$$\overline{\alpha}(Q^2) = (1.0 \sim 1.5)\frac{\Lambda^2_{QCD}}{Q^2}. \tag{7.16}$$

This is just a qualitative estimation of the running coupling constant in the full QCD theory in low momentum transfer. A more accurate result may be obtained by exactly solving the β-function of eq.(7.10). The running coupling constant in high momentum transfer is given in the usual perturbative QCD theory. A light-front perturbative QCD calculation of the leading order running coupling constant can be found from Ref. [14]. The result is standard:

$$\overline{\alpha}(Q^2) = \frac{\alpha_s(\mu^2)}{1 + b_0\alpha_s(\mu^2)\ln Q^2/\mu^2} = \frac{12\pi}{(33 - 2N_f)\ln Q^2/\Lambda^2_{QCD}}. \tag{7.17}$$

Up to date, no one precisely knows how the QCD coupling constant varies in low energy scale. However, it is interesting to see that the running coupling constant given by eqs.(7.13) and (7.17) for the small and large Q^2 respectively is indeed the basic assumption of the Richardson $Q\overline{Q}$ potential [30]:

$$V(Q^2) = -C_f\frac{\overline{\alpha}(Q^2)}{Q^2}, \tag{7.18}$$

where

$$\overline{\alpha}(Q^2) = \frac{12\pi}{(33 - 2N_f)\ln(1 + Q^2/\Lambda^2_{QCD})}. \tag{7.19}$$

The Richardson $Q\overline{Q}$ potential is proposed to exhibit the asymptotic freedom of QCD in short distance and a linear potential in large distance. From eq.(7.18), we see that for large Q^2 ($Q^2 >> \Lambda^2_{QCD}$),

$$\overline{\alpha}(Q^2) \sim \frac{12\pi}{(33 - 2N_f)\ln Q^2/\Lambda^2_{QCD}}, \tag{7.20}$$

which reproduces the result of the asymptotic freedom of QCD. The corresponding potential is just the Coulomb potential. For small Q^2 ($Q^2 << \Lambda^2_{QCD}$),

$$\overline{\alpha}(Q^2) \sim \frac{12\pi}{33 - 2N_f}\ \frac{\Lambda^2_{QCD}}{Q^2}, \tag{7.21}$$

and the corresponding potential from (7.18) becomes

$$V(Q^2) \sim -\frac{12\pi C_f\Lambda^2_{QCD}}{33 - 2N_f}\frac{1}{Q^4} \tag{7.22}$$

which is a Fourier transformation of the linear potential,

$$V(r) = kr. \tag{7.23}$$

The Fourier transformation of (7.18) is the $Q\overline{Q}$ potential in coordinate space:

$$V(r) = \frac{8\pi}{33 - 2N_f}\Lambda_{QCD}\left(r\Lambda_{QCD} - \frac{f(r\Lambda_{QCD})}{r\Lambda_{QCD}}\right), \tag{7.24}$$

where $f(t) = \left[1 - 4\int_1^\infty \frac{dq}{q}\frac{e^{-qt}}{[\ln(q^2-1)]^2+\pi^2}\right]$ [30]. The Richardson potential has successfully been used to describe quarkonium dynamics.

Comparing with eqs.(7.13) and (7.21), we have from the Richardson $Q\overline{Q}$ potential (with $N_f = 3$ [30])

$$c_0 = \frac{12\pi}{(33-2N_f)} = 1.4 \ . \tag{7.25}$$

This result agrees very well with eq.(7.16). Consequently, although it is a very rough qualitative analysis, the above result may imply that the confining Hamiltonian derived from light-front HQET in the similarity renormalization group scheme has also covered the dynamics of linear confining potential. As we have known, phenomenological potential quark models based on the Richardson potential, the linear plus Coulomb potential (also called the Cornell potential) as well as the logarithmic potential all give a good description of quarkonium dynamics [31]. Hence, it should be not surprising if our QCD confining Hamiltonian encompass the dynamic behavior of all these potentials.

VIII. DISCUSSIONS ON THE FULL QCD VIA EFFECTIVE THEORY

Thus far, the main ideas of the weak-coupling treatment on nonperturbative QCD proposed in the recent publication [1] have, at least qualitatively, been achieved for heavy quarkonium. The low energy nonperturbative QCD theory is defined by the effective low energy Hamiltonian H_λ. The key to solve this theory is to determine from the bound state equation the λ-dependence of the effective coupling constant α_λ. Our result indicates that the low energy effective QCD Hamiltonian exhibits an alternative realization of the inverse strong coupling expansion of the full QCD theory. Thus, with a suitable choice of the cutoff λ, the effective coupling constant α_λ (as well as g_λ) can be arbitrarily small. As an example, one may take λ to be a constituent gluon mass (about a half of the glueball masses, such as the recent possible evidences of $f_0(1500)$ and $\xi(2230)$ [23]),

$$\lambda : \ (0.75 \sim 1.5) \text{ GeV}. \tag{8.1}$$

Then the effective coupling constant (with $\overline{\Lambda} = 200$ MeV) is

$$\alpha_e(\lambda^2) = 0.06 \sim 0.01 \tag{8.2}$$

which is very small. The residual interaction $H_{\lambda I}$ is expanded in terms of this small coupling constant so that the corrections from $H_{\lambda I}$ can be perturbatively computed, and the weak-coupling treatment of nonperturbative QCD is explicitly realized.

Now the question is in what limit the effective theory can return back to the full theory of QCD so we can ensure that the present formulation is a complete consistent theory describing low energy QCD dynamics. This question is also important in the sense that with the cutoffs being introduced and the assumption of gluon being massive, the gauge symmetry and rotational symmetry may be broken down in the effective theory. Then we must know when all these symmetries can be restored. In ref. [1], we have argued that when $g_\lambda \to g_s$, the effective theory must recover the full QCD dynamics so that all symmetries are restored as well. However, it is not clear how this limit can be achieved. Here we shall attempt to answer this question from the heavy quarkonium solution.

We have used three cutoffs in this paper: the UV cutoff Λ, the low energy cutoff λ and the infrared longitudinal momentum cutoff ϵ. The UV cutoff is renormalized away in ordinary perturbation theory so that there is no any explicit Λ dependence in our formulation. The longitudinal infrared cutoff is automatically cancelled in all the physical sectors, due mainly to the gauge invariance as we have seen from the calculations throughout this work. The final formulation only contains the low energy cutoff λ. This λ dependence is essentially associated with nonperturbative QCD dynamics. However, the similarity renormalization group invariance on physical observations allows us to further remove away the λ dependence in physical sectors. We may define that the naive $\lambda \to 0$ limit brings the effective theory back to the full theory of QCD.

How can the limit of $\lambda \to 0$ theoretically bring the effective theory back to the full QCD theory? Firstly, recall that introducing the low energy cutoff in the effective theory is based on the assumption that gluon is massive in the low energy domain. Then the limit of $\lambda \to 0$ is

only allowed when the gluon mass goes to zero. With this limit the broken gauge symmetry due to the massive gluons is now restored. Secondly, once the cutoff λ is renormalized away, there is no any explicit cutoff dependence in the theory. Therefore the broken Lorentz symmetry due to the use of the explicit cutoff must be restored as well. Furthermore, the limit $\lambda \to 0$ corresponds to $Q^2 \to \infty$, where all gluons and quarks become current ones. Once all symmetries are restored and the current picture reemerges, the resulting theory should be the full QCD theory.

We may first check what happens if we perform the same procedure to QED for positronium. With $\lambda \to 0$, the similarity renormalization approach leads to an effective QED Hamiltonian in which the nonperturbative part $H_{\lambda 0}$ only contains the Coulomb interaction, and the remaining is the radiative correction $H_{\lambda I}$. No confining interactions and no infrared divergences occur, as expected. This is just the full QED theory used in the description of QED bound states. We can explicitly examine the above conclusion from eq.(6.23). To do so, we may first assume that the QED coupling constant is almost λ-independent because it is always very weak in the whole range of energy scale. Thus, with $\lambda \to 0$, we have from eq.(6.23),

$$\mathcal{E}_{nonc.} = 0, \quad \mathcal{E}_{Coul.} = -\frac{g^2}{\pi^2} C_f \sqrt{\pi} \omega \overline{\Lambda}, \tag{8.3}$$

namely, only Coulomb force contributes, and the confining interaction disappears, while the infrared divergence in the self-energy correction does not occur for $\lambda = 0$. Combining with the kinetic energy (where the term $\overline{\Lambda}^3/m_Q$ in $\mathcal{E}_{kin}$ is very small so that it can be neglected), the totally binding energy is given by

$$\overline{\Lambda} = \frac{3}{2m_Q}\omega^2 - \frac{g^2}{2\pi^2} C_f \sqrt{\pi}\omega. \tag{8.4}$$

Minimizing $\overline{\Lambda}$ with respect of ω, we obtain

$$\omega_0 = \frac{g^2 m_Q}{6\pi^{3/2}} C_f \ , \quad \overline{\Lambda} = -\frac{g^4 m_Q}{24\pi^3} C_f^2. \tag{8.5}$$

The above result can be rewritten as an exact solution of QED for the positronium ground

state in the nonrelativistic limit. If we let the color factor $C_f = 1$ and $m_Q \to m_e$, $g \to e$, eq.(8.5) can be reexpressed in terms of the Bohr radius and Bohr energy (in unit $h = 1$):

$$a_0 = \frac{\sqrt{\pi}}{2\omega_0} = \frac{3\pi^2}{m_e e^2} \ , \quad E_0 = \overline{\Lambda} = -\frac{\alpha}{2a_0} \ , \qquad (\alpha = \frac{e^2}{4\pi}) \tag{8.6}$$

where the Bohr radius is redefined since we use a Gaussian-type trial wavefunction which is not the same as the exact hydrogen atomic ground state wavefunction.

In fact, if we did not assume the λ-independent of the QED coupling constant, we would obtain, except for a color factor, the same similarity renormalization group β function of eq.(7.10). Thus, the fixed point of QED running coupling constant, $e = 0$, becomes infrared unstable, which is inconsistent with the well-known perturbative QED result of being an infrared stable fixed point. It implies that the similarity renormalization scheme can be applied to QED only for $\lambda = 0$. With $\lambda = 0$, our formulation reproduces the well-known method for solving QED bound states, namely, the nonperturbative bound states is determined by solving the Schrödinger equation with the Coulomb potential (because the effective Hamiltonian $H_{\lambda 0}$ only contains the Coulomb interaction at $\lambda = 0$) and the remaining relativistic radiative corrections (described here by $H_{\lambda I}$) can then be systematically computed in perturbation theory. In other words, the limit $\lambda \to 0$ brings the effective theory back to the full theory in QED.

However, unlike the QED, we cannot assume that the coupling constant g in QCD is independent of the scale λ. In other words, one cannot freely take the limit of $\lambda \to 0$. With λ being decreased, g_λ becomes larger and larger. Thus, the resulting $H_{\lambda 0}$ containing the Coulomb interaction alone is not sufficient to describe QCD bound states since the corrections from $H_{\lambda I}$ cannot be handled perturbatively. Therefore we are in practice unable to write down the full QCD theory in the weak-coupling formulation. The weak-coupling treatment of QCD in the limit $\lambda \to 0$ is no longer valid. Indeed, we find that with a small λ ($\lambda < 0.4$ GeV), eq.(6.23) has no solution. In other words, the so-called full QCD theory with massless gluon in low energy domain may only be formally interesting. To reproduce the acceptable hadronic properties with a small λ value, we must include more complicated

higher order contributions from $H_{\lambda I}$ into the nonperturbative bound state equation. Since the lowest bound value of λ is the constituent gluon mass, a finite λ, namely a nonzero constituent gluon mass, has effectively moved the nonperturbative contribution in the higher order processes into the low energy two-body confining interaction. The fact that the confining interaction dominates the binding dynamics of the quarkonium bound states at a finite λ (about 1 GeV) is indeed an evidence why nonperturbative QCD can be treated as a weak-couple problem in the present formulation. The limit $\lambda \to 0$ that can bring the effective theory back to the full theory is only implicative in QCD.

On the other hand, the result from quarkonium ground states seems to tell us that with the larger λ, the smaller the effective coupling constant can be reached. But this does not imply that the weak-coupling treatment to nonperturbative QCD works better for a larger λ. The scale dependence of the wavefunction provides a restriction on the value of λ. For quarkonia, ω_λ is decreased with increasing λ. However, ω_λ is proportional to the mean value of the (transverse) momentum square of the quark inside the heavy quarkonia which characterizes the size of hadrons. Therefore it should not be too large or too small in the best description for bound states. For the range of eq.(8.1), we have,

$$\omega_{\lambda 0} = 0.24 \sim 0.2 \text{ GeV}. \tag{8.7}$$

Correspondingly,

$$\langle r \rangle \sim \frac{1}{\omega_{\lambda 0}} = 0.8 \sim 1.0 \text{ fm}, \tag{8.8}$$

which gives a resonable quarkonium size. Therefore, the true low energy QCD theory is determined by H_λ with λ being around the hadronic mass scale.

Although our formulation thus explicitly involves λ, by the requirement of the similarity renormalization group invariance, all the physical observables computed in this effective theory can still be λ independent. The naive limit of $\lambda \to 0$ that brings the effective theory back to the full QCD theory may imply that the final physical results calculated from H_λ could also be gauge and Lorentz invariant, although H_λ itself does not have these symmetries (consequently the hadronic wavefunctions must also be λ-dependent).

In conclusion, the weak-coupling treatment approach to hadronic bound states in QCD can work well with the cutoff λ being around the hadronic mass scale. The well-defined bound state approach in QED is a special case ($\lambda = 0$) of the similarity renormalization group approach. The whole idea of the weak-coupling treatment to nonperturbative QCD via the similarity renormalization group approach is originally motivated from the bound state description of QED [1]. Now, a consistent connection between QCD and QED and their differences in low energy domain is explicitly examined.

IX. SUMMARY

In this last section, we shall summarize the general formulation and the main results we have obtained, and then briefly discuss the further works.

To realize the weak-coupling treatment of nonperturbative QCD recently proposed by Wilson and his collaborators [1], we have studied explicitly the heavy quark bound state problem, based on the recently developed light-front heavy quark effective theory of QCD [5,6]. Firstly, we have used the similarity renormalization group approach [1,2] to derive the effective confining Hamiltonian in the low energy scale for heavy quarks in heavy mass limit. To make the similarity renormalization approach practically manable, we have introduced a local cutoff scheme (2.4) to the bare QCD (and the effective heavy quark) Hamiltonian, which simplifies the cutoff scheme in [1]. Meanwhile we have also introduced a simple smearing function $f_{\lambda ij}$ (2.6) to the similarity renormalization group approach which further simplifies the original formulation of [1]. The resulting low-energy effective QCD Hamiltonian of heavy quark interactions exhibits the coexistence of a confining interaction and a Coulomb interaction on the light-front.

The realization of the weak-coupling treatment to nonperturbative QCD dynamics is very much based on the reseparation of the low energy effective Hamiltonian H_λ into a nonperturbative part $H_{\lambda 0}$ and the remaining as a perturbative term $H_{\lambda I}$, and also on the use of the constituent picture, as we have seen throughout the present work. The use of the

constituent picture in light-front field theory allows us to expand the heavy hadrons only with the valence quark Fock space. The light-front heavy quark effective theory also largely simplifies the structure of the heavy hadron bound state equations [see (4.26) and (4.31)].

A true realization of the weak-coupling approach to nonperturbative QCD can be obtained after solving the light-front bound state equations, from which one can determine the λ-dependence of the effective coupling constant in H_λ, as a solution of the similarity renormalization group invariance. We have used a well-behaved light-front wavefunction ansatz (a Gaussian form) to analytically solve the quarkonium bound state equation and determine the scale-dependence of the effective coupling constant. We have also shown that the effective coupling constant at the hadronic scale λ can be arbitrarily small. Thus, the possible weak-coupling treatment to nonperturbative QCD proposed in ref. [1] is explicitly achieved.

The results obtained in this paper is very optimistic. First, the λ-dependence of the effective coupling constant determines the similarity renormalization group β function, from which some qualitative running behavior of the coupling constant in low energy domain is obtained. The running coupling constant (7.16) is qualitatively the same one used in the successful Richardson $Q\overline{Q}$ potential for small momentum transfer (large distance). The later is a basic assumption to ensure the existence of a linear confining $Q\overline{Q}$ potential in large distance, which is now obtained from QCD in the present work. A light-front picture of quark confinement from QCD is naturally manifested in $H_{\lambda 0}$. It encompasses the general properties of these phenomenological confining potentials, the Richardson potential, the linear plus Coulomb potential and the logarithmic potentials used in the phenomenological description of quarkonium dynamics.

The weak-coupling treatment can be realized for nonperturbative QCD because the similarity renormalization group approach with a finite λ has extracted the confining interaction from the higher order nontrivial quark-gluon interactions into $H_{\lambda 0}$. Equivalently speaking, the similarity renormalization group approach implicatively provides an inverse strong interaction expansion of QCD via the low energy cutoff scale λ. As a physical consequence,

the confining interaction plays a dominant role in hadronic bound states, as we have seen in Section VII. The weak-coupling treatment of nonperturbative QCD is manable for λ being around the hadronic mass scale. The similarity renormalization group invariance can remove the λ-dependence in all the physical observables obtained from the effective H_λ. The possible connection of the effective theory to the full QCD theory in the limit $\lambda \to 0$ may further imply that the final physical results obtained from H_λ could be gauge and Lorentz invariant. Meanwhile, we have also shown that in consistence with the behavior of the QED β function, the similarity renormalization approach can be applied to QED only if $\lambda = 0$. Therefore, the confining picture obtained for QCD does not exist in QED but the well-known QED bound state method is reproduced. Thus, the consistency of the weak-coupling formulation has been qualitatively examined.

The applications of the present theory to heavy hadron spectroscopy and various heavy hadron decay processes can be simply achieved by numerically solving the bound state equations (4.26) and (4.31), and by further including the $1/m_Q$ corrections (which naturally leads to the spin splitting interactions). These will be presented in a forthcoming paper [28]. Finally, we should analyze the systematical approximations used in the whole computations in this paper, and then discuss the further works along this direction.

The entire derivations presented in this paper are purely based on the first principle of QCD. Applying to heavy hadrons, we took the heavy mass limit so that the QCD is reduced to HQET but the first order kinetic energy has been also included in the leading order Hamiltonian for heavy quarkonia. In the forthcoming paper [28], the $1/m_Q$ corrections (which contains all the spin splitting interactions) will be considered in the effective Hamiltonian $H_{\lambda I}$ in order to compute the heavy quarkonium spectroscopy and the next order correction to the bound states. These corrections should not affect on the main conclusions obtained in this paper since the small factor $1/m_Q$ plus the weak coupling constant guarantee that these $1/m_Q$ corrections can be computed perturbatively with respect to the nonperturbative heavy hadron bound states.

The low energy QCD Hamiltonian H_λ is obtained by a similarity renormalization trans-

formation to the bare QCD Hamiltonian (bare HQET Hamiltonian for heavy quarks). With the idea of the weak-coupling treatment to low energy QCD dynamics, the nonperturbative part $H_{\lambda 0}$ is computed upon to the second order in H_λ and the bound states are truncated to only including the valence quark Fock space. The higher order corrections in H_λ (included both the $1/m_Q$ corrections and the radiative corrections) can be examined in the usual Hamiltonian perturbation theory [28], which will also provide a consistent check to the validity of the present weak-coupling treatment. With these corrections computed in the Hamiltonian formulation, the contributions from the higher Fock space will be naturally included.

Finally, instead of numerically solving the light-front bound state equations, we used here a well-behaved light-front wavefunction ansatz to determine the bound state equation for heavy quarkonia. A numerical calculation of the light-front bound state equations (4.26) and (4.31) for the heavy hadronic spectroscopy will be also presented in [28]. Such a numerical computation of the bound states is much simpler in comparison to the lattice QCD simulation [8], and it will also directly give the hadronic wavefunctions in physical space. The resulting wavefunctions of the quarkonium bound states combining with the systematic computations of the subsequent radiative and $1/m_Q$ corrections thus provide a truly unified first- principles QCD description for various heavy quarkonium annihilation and production processes.

All the derivations presented in this paper are rigorous QCD derivation in low energy domain. The extension of the computations to heavy-light quark systems is straightforward but is more attractive in the current investigations on heavy hadrons. The extension of the present work to light-light hadrons requires a understanding of the chiral symmetry breaking mechanism in QCD which is a new challenge to nonperturbative QCD on the light-front. Nevertheless, the present work has provided a preliminary realization to the weak-coupling treatment of nonperturbative QCD proposed recently by Wilson et al. [1]. The new research along this direction is now in progress.

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FIGURES

FIG. 1. A plot of the confining potential energy, the Coulomb potential energy and the total potential energy as functions of the dimensionless variable ϖ but ϖ is proportional to $\sim r_l$ via ω_λ. The energies are scaled by the factor $\frac{g_\lambda^2 \lambda^2}{2\pi^2} C_f$.

FIG. 2. The λ-dependence of the effective coupling constant α_λ. The solid line is given by the analytical result (7.5), and the dots are obtained by numerically minimizing the quarkonium binding energy (6.23). Here λ is given in units of GeV.

FIG. 3. The λ-dependence of the wavefunction parameter $\omega_{\lambda 0}$ which is the solution of minimizing the quarkonium binding energy (6.23), where λ is given in units of GeV.

TABLES

Table I. Solution for charmonium ground state with $m_c = 1.4$ GeV

$\overline{\Lambda}$ (GeV)	α_λ	$\omega_{\lambda 0}$ (GeV)	$\mathcal{E}_{kin}(\mathrm{GeV}^2) + \mathcal{E}_{pot}(\mathrm{GeV}^2) = 2\overline{\Lambda}^2(\mathrm{GeV}^2)$		
0.2	0.02665	0.222	0.026836	0.053173	0.080009
0.3	0.06480	0.275	0.067902	0.112018	0.179920
0.4	0.11831	0.314	0.130225	0.189781	0.320006

Table II. Solution for bottomonium ground state with $m_b = 4.8$ GeV

$\overline{\Lambda}$ (GeV)	α_λ	$\omega_{\lambda 0}$ (GeV)	$\mathcal{E}_{kin}(\mathrm{GeV}^2) + \mathcal{E}_{pot}(\mathrm{GeV}^2) = 2\overline{\Lambda}^2(\mathrm{GeV}^2)$		
0.15	0.029965	0.492	0.023397	0.021602	0.044999
0.20	0.06795	0.623	0.050183	0.029816	0.079999
0.25	0.1385	0.779	0.098074	0.026930	0.125004

Table III. Some numerical solution on the λ-dependence of the weak coupling constant α_λ.

	charmonium		bottomonium	
λ (GeV)	α_λ	$\omega_{\lambda 0}$ (GeV)	α_λ	$\omega_{\lambda 0}$ (GeV)
0.75	0.05960	0.241		
1.0	0.02665	0.222	0.06795	0.623
1.5	0.00912	0.199	0.01607	0.478
2.0	0.00441	0.185	0.00695	0.427

Table IV. The λ-dependence of various interactions to the binding energy

λ (GeV)	0.5	0.75	1.0	1.2	1.4	1.8	2.0	3.0
$\mathcal{E}_{kin}$ (GeV^2)	0.04	0.031	0.027	0.025	0.023	0.021	0.02	0.018
$\mathcal{E}_{conf}$ (GeV^2)	0.049	0.050	0.053	0.055	0.057	0.059	0.06	0.062
$\mathcal{E}_{Col}$ (GeV^2)	-0.009	-0.001	-0.001	0.0	0.0	0.0	0.0	0.0

Fig.2

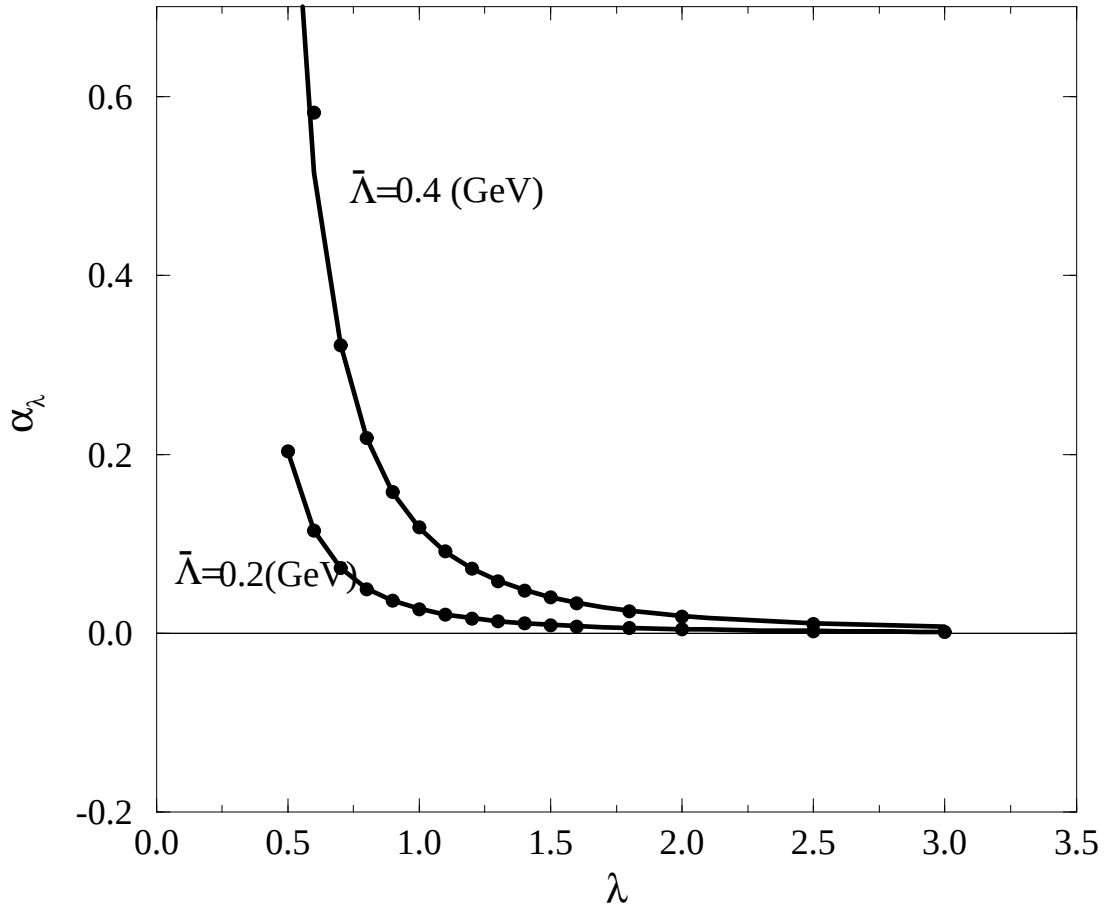
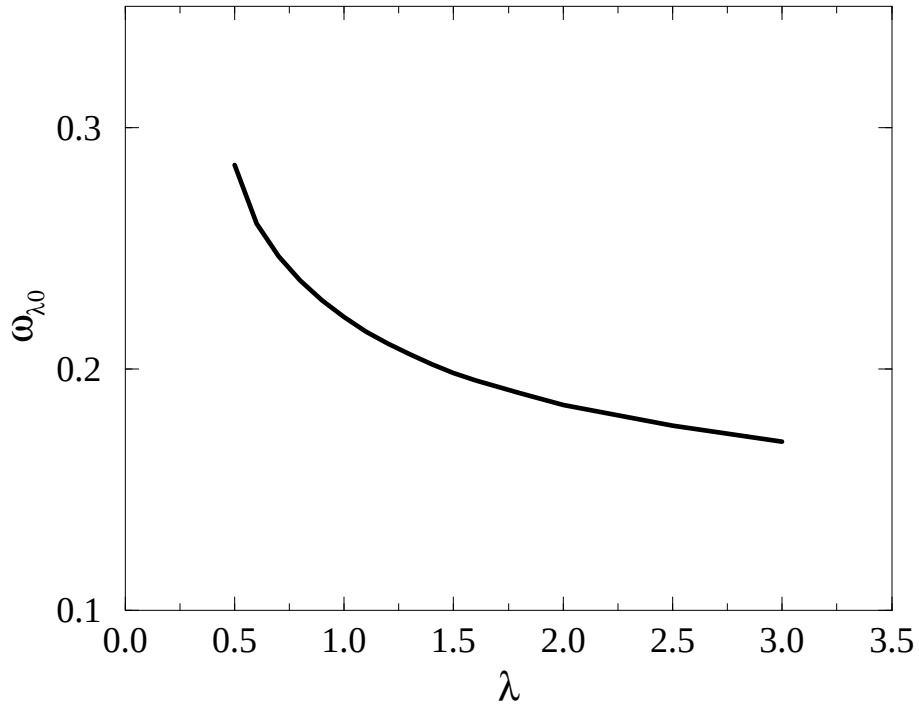

0.6
0.4
0.2
0.0
-0.2
α_λ
$\bar{\Lambda}$=0.4 (GeV)
$\bar{\Lambda}$=0.2(GeV)
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
λ

Fig.3



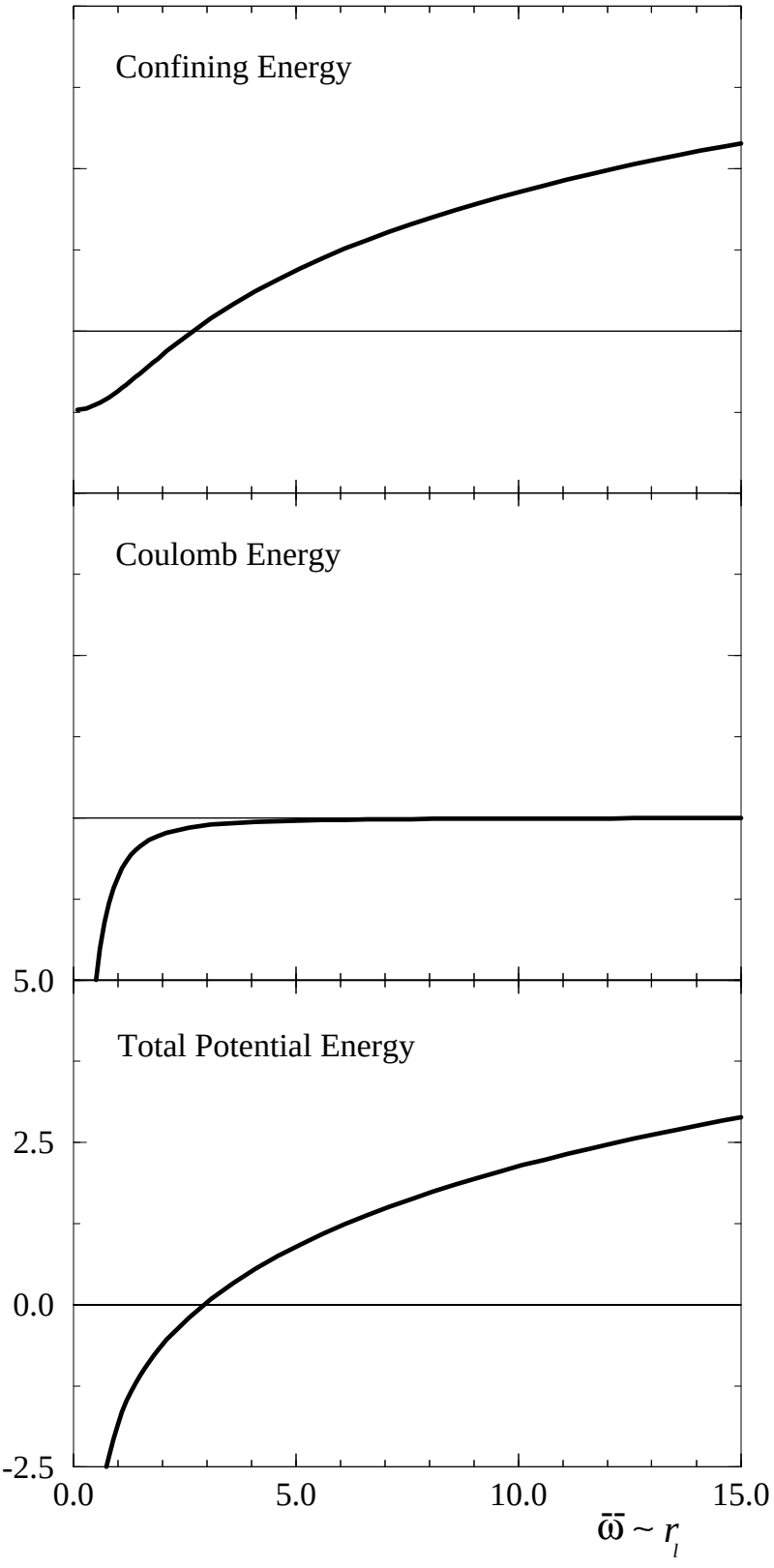

Confining Energy
Coulomb Energy
Total Potential Energy
5.0
2.5
0.0
-2.5
0.0
5.0
10.0
15.0
$\bar{\omega} \sim r_l$

Fig.1